The Schur Transformation for Generalized Nevanlinna Functions

Alpay, Daniel; Dijksma, Aad; Langer, Heinz; Shondin, Yuri

Published in:
Complex analysis and operator theory

DOI:
10.1007/s11785-006-0007-5

Publication date:
2007

Citation for published version (APA):
Alpay, D., Dijksma, A., Langer, H., & Shondin, Y. (2007). The Schur Transformation for Generalized Nevanlinna Functions: Interpolation and Self-Adjoint Operator Realizations. *Complex analysis and operator theory*, *1*(2), 169-210. https://doi.org/10.1007/s11785-006-0007-5

Compl. anal. oper. theory 1 (2007), 169–210

1661-8254/020169-42, *published online* December 23, 2006
DOI 10.1007/s11785-006-0007-5

Complex Analysis and Operator Theory

The Schur Transformation for Generalized Nevanlinna Functions: Interpolation and Self-Adjoint Operator Realizations

Daniel Alpay, Aad Dijksma, Heinz Langer, and Yuri Shondin

Abstract. The Schur transformation for generalized Nevanlinna functions has been defined and applied in [2]. In this paper we discuss its relation to a basic interpolation problem and study its effect on the minimal self-adjoint operator (or relation) realization of a generalized Nevanlinna function.

Mathematics Subject Classification (2000). 47A48, 47B50, 47B25, 47A06.

Keywords. Generalized Nevanlinna function, Schur transformation, Schur algorithm, interpolation, Pontryagin space, self-adjoint operator or relation realization.

1. Introduction

By $\mathbf{N}_0$ we denote the class of *Nevanlinna functions*: these are the functions $n(z)$ which have one of the following equivalent properties:

(i) $n(z)$ is locally holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies

$$\operatorname{Im} n(z) / \operatorname{Im} z \geq 0, \quad n(z^*) = n(z)^* \quad \text{if } z \in \mathbb{C} \setminus \mathbb{R}\,,$$

D. Alpay acknowledges with thanks the Earl Katz family for endowing the chair which supported this research and the Netherlands Organization for Scientific Research, NWO (grant B 61-524). The research of A. Dijksma and H. Langer was partly supported by the Center for Advanced Studies in Mathematics, CASM, of the Department of Mathematics of Ben-Gurion University, that of H. Langer also by the Austrian Science Fund, Project P15540-N05.

(ii) $n(z)$ is locally holomorphic on $\mathbb{C} \setminus \mathbb{R}$, $n(z^*) = n(z)^*$ if $z \in \mathbb{C} \setminus \mathbb{R}$, and the kernel

$$L_n(z,w) = \begin{cases} \dfrac{n(z) - n(w)^*}{z - w^*} & \text{if } z \neq w^* \,, \\ n'(z) & \text{if } z = w^* \end{cases}$$

is nonnegative on $\mathbb{C} \setminus \mathbb{R}$.

Let κ be an integer ≥ 0. By $\mathbf{N}_\kappa$ we denote the set of functions $n(z)$ which are locally meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and for which the kernel $L_n(z,w)$ has κ *negative squares on* $\operatorname{hol}(n)$; we also say that $n(z)$ has κ *negative squares* and write $\operatorname{sq}_-(n) = \kappa$. Here $\operatorname{hol}(n)$ is the *maximal domain of holomorphy of* $n(z)$ in $\mathbb{C}$, that is, the set of points from $\mathbb{C} \setminus \mathbb{R}$ at which $n(z)$ is holomorphic, and of those real points into which $n(z)$ can extended by holomorphy. We have $n(z) \in \mathbf{N}_\kappa$ if and only if

$$n(z) = \frac{\prod_{i=1}^{\kappa_1}(z-\alpha_i)(z-\alpha_i^*)}{\prod_{j=1}^{\kappa_2}(z-\beta_j)(z-\beta_j^*)}\, n_0(z)\,,$$

where $\alpha_i, \beta_j \in \mathbb{C}$, $\alpha_i \neq \beta_j$, $\max\{\kappa_1, \kappa_2\} = \kappa$, and $n_0(z)$ is a Nevanlinna function; see [5]. The elements of the set

$$\mathbf{N} = \bigcup_{\kappa \geq 0} \mathbf{N}_\kappa$$

are called *generalized Nevanlinna functions*.

The function $n(z)$ is a generalized Nevanlinna function if and only if it has a *self-adjoint realization* $\big(\mathcal{P}, A, \varphi(z)\big)$, in formula,

$$n(z) \sim \big(\mathcal{P}, A, \varphi(z)\big)\,, \tag{1.1}$$

which means that

(a) the *state space* $\mathcal{P}$ is a Pontryagin space,
(b) $A = A^*$ is a self-adjoint relation on $\mathcal{P}$ with nonempty resolvent set $\rho(A)$,
(c) $\varphi(z)$ is a corresponding *A-field*, that is, a locally holomorphic function on $\rho(A)$ with values in $\mathcal{P}$ satisfying

$$\varphi(z) = \big(I + (z-w)(A-z)^{-1}\big)\varphi(w), \quad z, w \in \rho(A)\,,$$

and

$$\big\langle \varphi(z), \varphi(w) \big\rangle_{\mathcal{P}} = L_n(z,w), \quad z, w \in \operatorname{hol}(n) \cap \rho(A)\,.$$

Note that if $n(z) \sim \big(\mathcal{P}, A, \varphi(z)\big)$ then also $n(z) + c \sim \big(\mathcal{P}, A, \varphi(z)\big)$ for any real number c. For a fixed point $z_0 \in \operatorname{hol}(n) \cap \rho(A)$ and $u_0 = \varphi(z_0)$ the realization (1.1) implies the equality

$$n(z) = n(z_0)^* + (z - z_0^*)\big\langle \big(I + (z-z_0)(A-z)^{-1}\big)u_0, u_0 \big\rangle_{\mathcal{P}}, \quad z \in \operatorname{hol}(n) \cap \rho(A)\,. \tag{1.2}$$

This formula we call a *self-adjoint realization of* $n(z)$ *centered at* z_0.

The self-adjoint realization $\big(\mathcal{P}, A, \varphi(z)\big)$ can always be chosen *minimal*, which means that

$$\overline{\operatorname{span}}\big\{\varphi(z) \mid z \in (\mathbb{C} \setminus \mathbb{R}) \cap \rho(A)\big\} = \mathcal{P}\,.$$

Minimality implies that the self-adjoint realization of $n(z)$ is unique up to unitary equivalence, and also that $\mathrm{hol}\,(n) = \rho(A)$ and $\mathrm{sq}_-(n) = \mathrm{ind}_-(\mathcal{P})$, where $\mathrm{ind}_-(\mathcal{P})$ is the negative index of the Pontryagin space $\mathcal{P}$; see [8].

An example of a minimal self-adjoint realization of a generalized Nevanlinna function $n(z)$ is the triple $\big(\mathcal{P}, A, \varphi(z)\big)$ where

(a) $\mathcal{P} = \mathcal{L}(n)$, the reproducing kernel Pontryagin space with kernel $L_n(z,w)$, whose elements are locally holomorphic functions $f(\zeta)$ on $\mathrm{hol}\,(n)$,

(b) A is the self-adjoint relation in $\mathcal{L}(n)$ with resolvent given by $(A-z)^{-1} = R_z$, the quotient difference operator defined by

$$(R_z f)(\zeta) = \begin{cases} \dfrac{f(\zeta)-f(z)}{\zeta - z} & \zeta \neq z, \\ f'(z) & \zeta = z, \end{cases} \quad f(\zeta) \in \mathcal{L}(n)\,,$$

(c) the A-field is given by $\varphi(z) = R_z n = L_n(\,\cdot\,, z^*)$, $\quad z \in \mathrm{hol}\,(n)$.

If $z_0 \in \mathrm{hol}\,(n)$, then also (1.2) holds with $u_0 = c\varphi(z_0)$ as well as with $u_0 = c\varphi(z_0^*)$, where c is a complex number of modulus 1.

For a generalized Nevanlinna function $n(z)$, not identically equal to a real constant, the Schur transform $\widehat{n}(z)$ with respect to some fixed point $z_1 \in \mathbb{C}$, which is also a generalized Nevanlinna function or the constant ∞, was defined in [2]. We recall the definition of the Schur transformation and some of the main results related to it from [2] in Section 2. In [2] we motivated the definition of the Schur transformation by showing that it can be applied to obtain minimal factorizations of rational J_ℓ-unitary 2×2-matrix functions, having the only pole at z_1, with elementary factors of the same type. In the present paper we give another motivation for the definition of the Schur transformation: we show that it can be used to solve the basic interpolation problem for generalized Nevanlinna functions. This interpolation problem, in its simplest form, can be formulated as follows: For $\nu_0 \in \mathbb{C}$ determine all $n(z) \in \mathbf{N}_\kappa$ which are holomorphic at z_1 and satisfy $n(z_1) = \nu_0$; see Section 3.

Another aim of the present paper is to study the effect of the Schur transformation on the minimal self-adjoint realizations: If $n(z)$ and its Schur transform $\widehat{n}(z)$ have the minimal self-adjoint realizations $\big(\mathcal{P}, A, \varphi(z)\big)$ and $\big(\widehat{\mathcal{P}}, \widehat{A}, \widehat{\varphi}(z)\big)$, respectively, we want to know how the latter can be obtained from the first; see Sections 4 and 5. In Section 6 we consider the composite Schur transform, which consists possibly of two steps in order to stay in the class of functions holomorphic at z_1. In these sections we explain the geometric meaning of the Schur transformation: it corresponds to a restriction to a subspace of the state space and to the compression of the operator or relation to this subspace. In our opinion this yields another motivation for this form of the Schur transformation. In Section 7 we show, using our realization results, that if the Schur algorithm can be applied to a function $n(z) \in \mathbf{N}$ infinitely often, that is, if the sequence $(n_j(z))_{j\geq 0}$ with $n_0(z) = n(z)$ and $n_j(z) = \widehat{n}_{j-1}(z)$ is well defined, then $n_j(z) \in \mathbf{N}_0$ for all sufficiently large j. Finally, in Section 8 we relate the results of Sections 4, 5 and 6 to the explicit minimal self-adjoint realization of $n(z)$ in the reproducing kernel Pontryagin space with kernel $L_n(z,w)$ as described above.

2. Definition of the Schur transformation $\widehat{n}(z)$

Throughout the paper the point $z_1 \in \mathbb{C}^+$ is fixed. If $n(z) \in \mathbf{N}$ is not identically equal to a real constant, we define its *Schur transform* $\widehat{n}(z)$ (relative to z_1) as follows. There are three cases to consider and to each case we add some remarks which were proved in [2] and will be used in the sequel. In Case I and Case III the function $n(z)$ is holomorphic at z_1 and $\operatorname{Im} n(z_1) \neq 0$ and $\operatorname{Im} n(z_1) = 0$, respectively, in Case II the function $n(z)$ has a pole at z_1 (and hence also at z_1^*). In Cases I and III we denote by $\nu_j \in \mathbb{C}$, $j = 0, 1, \dots$, the Taylor coefficients of $n(z)$ at z_1:

$$n(z) = \sum_{j=0}^{\infty} \nu_j (z - z_1)^j\,; \tag{2.1}$$

this relation holds in a neighborhood of z_1, which we need not specify. We set

$$\mu = \frac{\nu_0 - \nu_0^*}{z_1 - z_1^*}\,.$$

Definition Schur transformation Case I. If $n(z)$ is holomorphic at z_1 with Taylor expansion (2.1) and if Im $\nu_0 \neq 0$, then $\widehat{n}(z) = \infty$ if $n(z)$ is linear and otherwise

$$\widehat{n}(z) = \frac{\beta(z)n(z) - |\nu_0|^2}{n(z) - \alpha(z)}\,, \tag{2.2}$$

where

$$\alpha(z) = \nu_0 + \mu(z - z_1), \quad \beta(z) = \nu_0^* - \mu(z - z_1)\,.$$

As an example we mention the case where $n(z) = \nu_0$, $\operatorname{Im} z > 0$ (hence $n(z) = \nu_0^*$, $\operatorname{Im} z < 0$), then $\operatorname{Im} \nu_0 > 0$, otherwise $n(z)$ is not a generalized Nevanlinna function, and $\widehat{n}(z) = n(z)$. In fact, the latter equality holds if and only if $n(z)$ is a Nevanlinna function which is a constant with positive imaginary part on the upper half plane $\mathbb{C}^+$.

Remarks 2.1. In the following remarks we assume $\operatorname{Im} \nu_0 \neq 0$.

(1) If $n(z)$ is linear, that is, of the form $n(z) = a + bz$ with $a, b \in \mathbb{R}$, then

$$n(z) = \nu_0 + \mu(z - z_1) = \alpha(z)\,.$$

In this case the denominator of (2.2) vanishes and this is in accordance with the fact that we have set $\widehat{n}(z) = \infty$.

(2) We have

$$\beta(z)n(z) - |\nu_0|^2 = (\nu_0^*\nu_1 - \nu_0\mu)(z - z_1) + \sum_{j=2}^{\infty}(\nu_0^*\nu_j - \nu_{j-1}\mu)(z - z_1)^j\,, \tag{2.3}$$

$$n(z) - \alpha(z) = (\nu_1 - \mu)(z - z_1) + \sum_{j=2}^{\infty}\nu_j(z - z_1)^j\,.$$

Hence, if $n(z)$ is not linear, that is, $n(z) \not\equiv \alpha(z)$, the function $\widehat{n}(z)$ in (2.2) is holomorphic at z_1 if and only if

$$\nu_1 \neq \mu\,. \tag{2.4}$$

The sufficiency of this condition is clear. On the other hand, $\nu_1 = \mu$ implies that the first coefficient on the righthand side of (2.3) is $\neq 0$ since $\operatorname{Im}\nu_0 \neq 0$, and therefore $\widehat{n}(z)$ has a pole at z_1 whose order $q \geq 1$ is equal to the order of the zero z_1 of $n(z) - \alpha(z)$ minus 1, that is, q can be obtained from the relations

$$\nu_1 = \mu, \quad \nu_2 = \cdots = \nu_q = 0,\ \nu_{q+1} \neq 0\,.$$

Since we assume that $n(z)$ is not linear such an integer $q \geq 1$ exists.

(3) If $n(z) \in \mathbf{N}_\kappa$ is not linear, then $\widehat{n}(z) \in \mathbf{N}_{\widehat{\kappa}}$ with $\widehat{\kappa} = \kappa$ if Im $\nu_0 > 0$ and $\widehat{\kappa} = \kappa - 1$ if Im $\nu_0 < 0$; see Section 4.

Definition Schur transformation Case II. If $n(z)$ has a pole at z_1 then

$$\widehat{n}(z) = n(z) - h_{z_1}(z) - h_{z_1^*}(z)\,,$$

where $h_{z_1}(z)$ and $h_{z_1^*}(z)$ are the principal parts of the Laurent expansion of $n(z)$ at the points z_1 and z_1^* respectively.

Remarks 2.2. In the following remarks we assume $n(z)$ has a pole at z_1.

(1) We have $h_{z_1^*}(z) = h_{z_1}(z^*)^*$, because $n(z^*) = n(z)^*$. Hence, if the order of the pole at z_1 is $q \geq 1$, then

$$h_{z_1}(z) + h_{z_1^*}(z) = \frac{r(z)}{(z - z_1)^q(z - z_1^*)^q}\,,$$

where $r(z)$ is a polynomial of degree $\leq 2q - 1$, which is real in the sense that $r(z)^* = r(z^*)$ and satisfies $r(z_1) \neq 0$.

(2) If $n(z) \in \mathbf{N}_\kappa$ and $n(z)$ has a pole at z_1 of order $q \geq 1$ then $q \leq \kappa$ and $\widehat{n}(z) \in \mathbf{N}_{\kappa-q}$ and is holomorphic at z_1.

To define the Schur transformation in Case III ($n(z)$ holomorphic and with Taylor expansion (2.1) at z_1 and $\operatorname{Im}\nu_0 = 0$) we need some preparation. Clearly, since $n(z)$ is not a real constant, the function

$$\frac{1}{n(z) - \nu_0}$$

has poles at z_1 and z_1^* of order k, where k is the smallest integer ≥ 1 such that $\nu_k \neq 0$. If we denote the principal parts of this function at z_1 and z_1^* by $H_{z_1}(z)$ and $H_{z_1^*}(z)$, respectively, we can write

$$\frac{1}{n(z)-\nu_0} = H_{z_1}(z) + H_{z_1^*}(z) + a(z) = \frac{p(z)}{(z-z_1)^k(z-z_1^*)^k} + a(z)\,, \tag{2.5}$$

with a function $a(z)$ which is holomorphic at z_1 and a polynomial $p(z)$ which is real, of degree $\leq 2k-1$, and such that $p(z_1) \neq 0$. This polynomial (or actually the whole first term on the righthand side of (2.5)) will play an important role in the definition of the Schur transformation. In Section 3 it is crucial, that this polynomial $p(z)$ is determined by the coefficients $\nu_k, \nu_{k+1}, \ldots, \nu_{2k-1}$ of the Taylor expansion (2.1). To see this we write the relation (2.5) in the form

$$p(z)\big(n(z)-\nu_0\big) = (z-z_1)^k(z-z_1^*)^k + \mathrm{O}\big((z-z_1)^{2k}\big)\,.$$

With

$$p(z) = \sum_{j=0}^{k-1} a_j(z-z_1)^j + \sum_{j=k}^{2k-1} b_j(z-z_1)^j \tag{2.6}$$

we find

$$\begin{aligned}\left(\sum_{j=0}^{k-1} a_j(z-z_1)^j + \sum_{j=k}^{2k-1} b_j(z-z_1)^j\right)&\sum_{\ell=k}^{\infty} \nu_\ell(z-z_1)^\ell \\ &= (z-z_1)^k\big(z-z_1+(z_1-z_1^*)\big)^k + \mathrm{O}\big((z-z_1)^{2k}\big) \\ &= (z-z_1)^k\sum_{j=0}^{k}\binom{k}{j}(z-z_1)^j(z_1-z_1^*)^{k-j} + \mathrm{O}\big((z-z_1)^{2k}\big)\,,\end{aligned}$$

hence the coefficients $a_0, \ldots, a_{k-1}$ are determined by the relations

$$a_j\nu_k + a_{j-1}\nu_{k+1} + \cdots + a_0\nu_{k+j} = \binom{k}{j}(z_1-z_1^*)^{k-j}, \quad j = 0, 1, \ldots, k-1\,. \tag{2.7}$$

The coefficients $b_k, \ldots, b_{2k-1}$ are determined through the a_j by the fact that $p(z)$ is real: $p(z) = p(z^*)^*$. In [2] it is shown that they are the unique solutions of the system of k equations

$$\sum_{j=k}^{2k-1} b_j\frac{j!}{(j-i)!}(z_1^*-z_1)^{j-i} = i!a_i^* - \sum_{j=i}^{k-1} a_j\frac{j!}{(j-i)!}(z_1^*-z_1)^{j-i}\,, \quad i = 0, 1, \ldots, k-1\,.$$

Definition Schur transformation Case III. Now suppose that $n(z)$ is not a real constant, holomorphic with Taylor expansion (2.1) at z_1, and $\mathrm{Im}\ \nu_0 = 0$. Let $k \geq 1$ be the smallest integer such that $\nu_k \neq 0$. If

$$\frac{1}{n(z)-\nu_0}$$

has poles only at z_1 and z_1^* and vanishes at ∞, that is, if

$$n(z) = \nu_0 + \frac{(z-z_1)^k (z-z_1^*)^k}{p(z)}, \tag{2.8}$$

then $\widehat{n}(z) = \infty$. If $n(z)$ is not of this special form, we introduce the functions

$$\alpha(z) = \nu_0 + \frac{(z-z_1)^k (z-z_1^*)^k}{p(z)}, \quad \beta(z) = \nu_0 - \frac{(z-z_1)^k (z-z_1^*)^k}{p(z)},$$

and define

$$\widehat{n}(z) = \frac{\beta(z)n(z) - \nu_0^2}{n(z) - \alpha(z)}. \tag{2.9}$$

Remarks 2.3. We assume in this remark that $\operatorname{Im} \nu_0 = 0$ and that $n(z)$ is not of the form (2.8).

(1) For $z \to z_1$ and with a_k, b_k from (2.6) we have

$$\beta(z)n(z) - \nu_0^2 = \nu_k^2 \left(\nu_0 \frac{b_k - a_k}{(z_1 - z_1^*)^k} - 1 \right) (z - z_1)^{2k} + \mathrm{O}\left((z-z_1)^{2k+1} \right),$$

$$n(z) - \alpha(z) = \nu_k^2 \frac{b_k - a_k}{(z_1 - z_1^*)^k} (z - z_1)^{2k} + \mathrm{O}\left((z-z_1)^{2k+1} \right).$$

Hence $\widehat{n}(z)$ is holomorphic at z_1 if and only if $a_k \neq b_k$; otherwise it has a pole and the order of the pole is equal to the order of the zero at z_1 of $n(z) - \alpha(z)$ minus $2k$.

(2) If $n(z) \in \mathbf{N}_\kappa$ and $\widehat{n}(z)$ is given by (2.9), then $(1 \leq) k \leq \kappa$ and $\widehat{n}(z) \in \mathbf{N}_{\widehat{\kappa}}$ with $\widehat{\kappa} = \kappa - k$.

Example 2.4. Assume $\alpha, \beta, \gamma \in \mathbb{R}$ and $\gamma > 0$. The relation

$$n(z) = \alpha + \beta z + i\gamma, \quad \operatorname{Im} z > 0, \tag{2.10}$$

defines a function $n(z)$ in $\mathbf{N}_0$ if $\beta \geq 0$ and in $\mathbf{N}_1$ if $\beta < 0$. The case $\beta = 0$ was mentioned as an example after the definition of the Schur transformation Case I. The following considerations show that the class of functions of the form (2.10) is invariant under the Schur transformation. We consider two cases:

(i) $\operatorname{Im} n(z_1) = \beta \operatorname{Im} z_1 + \gamma \neq 0$. Then Case I of the definition of the Schur transformation applies and we have

$$\widehat{n}(z) = \alpha_1 + \beta_1 z + i\gamma_1, \quad \operatorname{Im} z > 0,$$

with

$$\alpha_1 = \alpha - \frac{\beta^2 \operatorname{Im} z_1 \operatorname{Re} z_1}{\gamma}, \quad \beta_1 = \beta \left(1 + \frac{\beta \operatorname{Im} z_1}{\gamma} \right), \quad \gamma_1 = \gamma \left(1 + \frac{\beta \operatorname{Im} z_1}{\gamma} \right)^2,$$

and

$$\operatorname{Im} \widehat{n}(z_1) = \gamma \left(1 + \frac{\beta \operatorname{Im} z_1}{\gamma} \right) \left(1 + 2\frac{\beta \operatorname{Im} z_1}{\gamma} \right).$$

If $\beta \geq 0$, then $\widehat{n}(z) \in \mathbf{N}_0$. If $\beta < 0$ and $0 < \gamma \leq -\beta \operatorname{Im} z_1$, then $\widehat{n}(z)$ belongs to $\mathbf{N}_0$; in case $\gamma = -\beta \operatorname{Im} z_1$ we have $\widehat{n}(z) \equiv \alpha_1$. If, on the other hand, $\beta < 0$ and $\gamma > -\beta \operatorname{Im} z_1$, then $\widehat{n}(z)$ belongs to $\mathbf{N}_1$.

(ii) $\operatorname{Im} n(z_1) = \beta \operatorname{Im} z_1 + \gamma = 0$. In this case $\beta < 0$. Now Case III of the definition of the Schur transformation applies and we have

$$\widehat{n}(z) = (\alpha + 2\beta \operatorname{Re} z_1) - \beta z + i\gamma, \quad \operatorname{Im} z > 0\,,$$

which belongs to $\mathbf{N}_0$ and $\operatorname{Im} \widehat{n}(z_1) = -\beta \operatorname{Im} z_1 + \gamma > 0$.

The definition of the Schur transformation given in this note differs slightly from the one given in [2]. In [2] we defined it only for Nevanlinna functions $n(z)$ which are holomorphic at z_1 and then so that $\widehat{n}(z)$ was again a generalized Nevanlinna function holomorphic at z_1. Thus in that paper the Schur transformation was defined by I and III composed with the Schur transformation II in case the resulting transform had a pole. In the last case we denote this composition of the two Schur transformations applied to $n(z)$ by $\widetilde{n}(z)$ and call it the *composite Schur transform*; see Section 6.

3. Basic interpolation problem

The basic interpolation problem at $z = z_1 \in \mathbb{C}^+$ for generalized Nevanlinna functions reads as follows.

Problem 3.1. *Given $\nu_0 \in \mathbb{C}$ and an integer $\kappa \geq 0$. Determine all $n(z) \in \mathbf{N}_\kappa$ which are holomorphic at z_1 and satisfy $n(z_1) = \nu_0$.*

To describe the solutions of this basic interpolation problem we consider two cases:

Case Im $\nu_0 \neq 0$. This case corresponds to Case I of the definition of the Schur transformation. If $\kappa = 0$ and Im $\nu_0 < 0$, then the problem does not have a solution, because $\operatorname{Im} n(z_1) \geq 0$ for all functions $n(z)$ in the class $\mathbf{N}_0$. If Im $\nu_0 > 0$, then for each $\kappa \geq 0$ and if Im $\nu_0 < 0$ then for each $\kappa \geq 1$ there are infinitely many solutions as the following theorem shows. To formulate it we use the notation introduced in Section 2:

$$\mu = \frac{\nu_0 - \nu_0^*}{z_1 - z_1^*}$$

and

$$\alpha(z) = \nu_0 + \mu(z - z_1), \quad \beta(z) = \nu_0^* - \mu(z - z_1)\,.$$

Theorem 3.2. *If* Im $\nu_0 \neq 0$, *the formula*

$$n(z) = \frac{\alpha(z)\widetilde{n}(z) - |\nu_0|^2}{\widetilde{n}(z) - \beta(z)} \tag{3.1}$$

gives a one-to-one correspondence between all solutions $n(z) \in \mathbf{N}_\kappa$ of Problem 3.1 *and all parameters $\widetilde{n}(z) \in \mathbf{N}_{\widetilde{\kappa}}$, which if holomorphic at z_1 satisfy the inequality $\widetilde{n}(z_1) \neq \nu_0^*$, where*

$$\widetilde{\kappa} = \begin{cases} \kappa & \text{if} \quad \operatorname{Im} \nu_0 > 0\,, \\ \kappa - 1 & \text{if} \quad \operatorname{Im} \nu_0 < 0\,. \end{cases}$$

The theorem will be proved later in this section. If $\kappa = 0$ and $\operatorname{Im}\nu_0 > 0$, then Problem 3.1 is the Nevanlinna–Pick interpolation problem at one point, and the parmetrization formula (3.1) is due to V. P. Potapov. Note that for all parameters $\widetilde{n}(z) \in \mathbf{N}$ which have a pole at z_1 the solution (3.1) satisfies

$$n'(z_1) = \mu\,.$$

This follows from the relation

$$n(z) - \nu_0 = \mu(z - z_1)\frac{\widetilde{n}(z) - \nu_0}{\widetilde{n}(z) - \beta(z)}\,. \tag{3.2}$$

If $\Theta(z)$ is a 2×2 matrix function,

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix},$$

then by $\mathcal{T}_{\Theta(z)}(f(z))$ we denote the *fractional linear transformation of the function* $f(z)$:

$$\mathcal{T}_{\Theta(z)}(f(z)) = \frac{a(z)f(z) + b(z)}{c(z)f(z) + d(z)}\,;$$

if $f(z) \equiv \infty$, the righthand side stands for the quotient $a(z)/c(z)$. Defining the 2×2 matrix J_ℓ and the function $b_\ell(z)$ by

$$J_\ell = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b_\ell(z) = \frac{z - z_1}{z - z_1^*}\,,$$

the parametrization formula (3.1) can be written as the fractional linear transformation

$$n(z) = \mathcal{T}_{\Theta_1(z)}\big(\widetilde{n}(z)\big)\,,$$

where

$$\Theta_1(z) = I_2 + \big(b_\ell(z) - 1\big)\frac{\mathbf{u}\mathbf{u}^* J_\ell}{\mathbf{u}^* J_\ell \mathbf{u}}, \quad \mathbf{u} = \begin{pmatrix} \nu_0^* \\ 1 \end{pmatrix}.$$

Case $\operatorname{Im}\nu_0 = 0$. This case corresponds to case III of the definition of the Schur transformation. By the maximum modulus principle, there is a unique solution in the class $\mathbf{N}_0$, namely $n(z) \equiv \nu_0$. There are infinitely many solutions in $\mathbf{N}_\kappa$ for $\kappa \geq 1$. To describe them we reformulate the problem with augmented parameters.

Problem 3.3. *Given $\nu_0 \in \mathbb{C}$ with $\operatorname{Im}\,\nu_0 = 0$, an integer κ with $\kappa \geq 1$, an integer k with $1 \leq k \leq \kappa$ and numbers $s_0, s_1, \ldots, s_{k-1} \in \mathbb{C}$ with $s_0 \neq 0$. Determine all functions $n(z) \in \mathbf{N}_\kappa$, which are holomorphic at z_1 and such that $n(z_1) = \nu_0$, $\nu_{k+j} = s_j$, $j = 0, 1, \ldots, k-1$, and, if $k > 1$, $\nu_1 = \cdots = \nu_{k-1} = 0$.*

With the data $\nu_0; s_0 \neq 0, s_1, \ldots, s_{k-1}$ we define a polynomial $p(z; s_0, \ldots, s_{k-1})$ and functions $\alpha(z), \beta(z)$ as follows. The polynomial $p(z; s_0, \ldots, s_{k-1})$ is of degree $\leq 2k - 1$ and has the properties:

(1) The numbers $a_j = p^{(j)}(z_1)/j!$, $j = 0, \ldots, k-1$, satisfy the relations

$$a_j s_0 + a_{j-1} s_1 + \cdots + a_0 s_j = \binom{k}{j}(z_1 - z_1^*)^{k-j}, \quad j = 0, 1, \ldots, k-1\,,$$

(2) $p(z)$ is real, that is, $p(z) = p(z^*)^*$.

That $p(z)$ is uniquely determined follows from considerations as after formula (2.6) in Section 2. Further we define

$$\alpha(z) = \nu_0 + \frac{(z - z_1)^k (z - z_1^*)^k}{p(z)}, \quad \beta(z) = \nu_0 - \frac{(z - z_1)^k (z - z_1^*)^k}{p(z)}\,.$$

Theorem 3.4. *Let $\nu_0 \in \mathbb{R}$ and an integer $\kappa \geq 1$ be given. For each integer k with $1 \leq k \leq \kappa$ and any choice of the complex numbers $s_0 \neq 0, s_1, \ldots, s_{k-1}$, the formula*

$$n(z) = \frac{\alpha(z)\widetilde{n}(z) - |\nu_0|^2}{\widetilde{n}(z) - \beta(z)}\,, \tag{3.3}$$

establishes a one-to-one correspondence between all solutions of the Problem 3.3 *and all parameters $\widetilde{n}(z) \in \mathbf{N}_{\kappa-k}$, such that $\widetilde{n}(z_1) \neq \nu_0$ if $\widetilde{n}(z)$ is holomorphic at z_1.*

The parametrization formula (3.3) can be written as the fractional linear transformation

$$n(z) = \mathcal{T}_{\Theta_2(z)}\big(\widetilde{n}(z)\big)\,,$$

where

$$\Theta_2(z) = I_2 - \frac{p(z)}{(z - z_1)^k (z - z_1^*)^k}\,\mathbf{u}\mathbf{u}^* J_\ell, \quad \mathbf{u} = \begin{pmatrix} \nu_0 \\ 1 \end{pmatrix}.$$

If $n(z) \in \mathbf{N}$ is holomorphic at z_1, then it is a solution of Problem 3.1 or Problem 3.3 with data determined by its own Taylor coefficients. Thus there exists a unique parameter $\widetilde{n}(z)$ such that the equality in (3.1) or (3.3) holds. This function is precisely the Schur transform of $n(z)$, that is, for $j = 1$ or $j = 2$, respectively,

$$\widehat{n}(z) = \widetilde{n}(z) = \mathcal{T}_{\Theta_j(z)^{-1}}\big(n(z)\big)\,.$$

We note that also $\widehat{n}(z)$ in Case II of the definition the Schur transformation can be expressed as

$$\widehat{n}(z) = \mathcal{T}_{\Theta_3(z)^{-1}}\big(n(z)\big)\,,$$

namely with

$$\Theta_3(z) = I_2 - \big(h_{z_1}(z) + h_{z_1^*}(z)\big)\mathbf{u}\mathbf{u}^* J_\ell, \quad \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

By $\mathcal{U}_\ell$ we denote the set of all rational 2×2 matrix functions $\Theta(z)$ which are J_ℓ-unitary on the line, that is, satisfy

$$\Theta(z) J_\ell \Theta(z)^* = J_\ell, \quad z \in \mathbb{R} \cap \operatorname{hol}(\Theta); \quad J_\ell = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

With $z_1 \in \mathbb{C}^+$, $\mathcal{U}_\ell^{z_1}$ stands for the set of those matrix functions from $\mathcal{U}_\ell$ which have a unique pole at z_1^*, and thus are of the form

$$\Theta(z) = \sum_{j=0}^{m} \frac{T_j}{(z - z_1^*)^j}\,, \tag{3.4}$$

where m is an integer ≥ 0 and T_j is a 2×2 matrix, $j = 0, 1, \ldots, m$. As examples we mention that

$$\Theta_1(z), \quad \Theta_2(z) b_\ell(z)^k, \quad \Theta_3(z) b_\ell(z)^k \in \mathcal{U}_\ell^{z_1}\,, \tag{3.5}$$

and that the constant matrices in this set are given by

$$\mathrm{e}^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with θ, a, b, c, $d \in \mathbb{R}$ such that $ad - bc = 1$. Note that the last two functions in (3.5) define the same fractional linear transformations as $\Theta_2(z)$ and $\Theta_3(z)$, respectively.

With $\Theta(z) \in \mathcal{U}_\ell$ we associate the kernel

$$K_\Theta(z, w) = \frac{J_\ell - \Theta(z) J_\ell \Theta(w)^*}{z - w^*}\,,$$

which has at most finitely many negative squares, and denote the corresponding reproducing kernel Pontryagin space by $\mathcal{P}(\Theta)$. The number of negative squares of the kernels associated with the functions in (3.5) is 0 if Im $\nu_0 > 0$ and 1 if Im $\nu_0 < 0$ for the first function and is equal to k for the other two functions.

The basic interpolation problem is one of the analytic motivations for the Schur transformation of Nevanlinna functions. Another one is the unique factorization of functions from the class $\mathcal{U}_\ell^{z_1}$ into elementary factors: in [2] it is shown that the elementary factors in $\mathcal{U}_\ell^{z_1}$ are precisely the ones in (3.5) up to multiplication by constant matrices in $\mathcal{U}_\ell$. Our results in the next sections show the geometrical features of the definition of the Schur transformation.

Proof of Theorem 3.2. If $n(z)$ is a solution of Problem 3.1, then (3.1) holds with $\widetilde{n}(z) = \widehat{n}(z)$, the Schur transform of $n(z)$. If $\widetilde{n}(z)$ is holomorphic at z_1, then by the formulas in Remark 2.1 (2), we have $\widetilde{n}(z_1) \neq \nu_0^*$. By Remark 2.1 (3), $\widetilde{n}(z) \in \mathbf{N}_{\widetilde{\kappa}}$. Let $\widetilde{n}(z) \in \mathbf{N}_{\widetilde{\kappa}}$ be a parameter with $\widetilde{n}(z_1) \neq \nu_0^*$ if holomorphic at z_1. Then $n(z)$ defined by (3.1) is a solution of Problem 3.1 and this follows from (3.2). □

Proof of Theorem 3.4. The proof of this theorem is like the proof of Theorem 3.2. If $n(z)$ is a solution of Problem 3.3, then (3.3) holds with $\widetilde{n}(z) = \widehat{n}(z)$, the Schur transform of $n(z)$. If $\widetilde{n}(z)$ is holomorphic at z_1, then by the formulas in Remark 2.3 (1), we have $\widetilde{n}(z_1) \neq \nu_0$. By Remark 2.3 (2), $\widetilde{n}(z) \in \mathbf{N}_{\kappa-k}$.

Let $\widetilde{n}(z) \in \mathbf{N}_{\kappa-k}$ be a parameter with $\widetilde{n}(z_1) \neq \nu_0$ if holomorphic at z_1 and let $n(z)$ be defined by (3.3). Rewriting (3.3) in the form

$$p(z)\big(n(z) - \nu_0\big) = (z - z_1)^k (z - z_1^*)^k - \frac{n(z) - \nu_0}{\widetilde{n}(z) - \nu_0}\,(z - z_1)^k (z - z_1^*)^k$$

and using $p(z_1) \neq 0$ we see that $n(z) - \nu_0$ has a zero at z_1 of order at least k and that therefore the second summand on the righthand side is $\mathrm{O}\big((z-z_1)^{2k}\big)$ as $z \to z_1$. Hence $\nu_1 = \cdots = \nu_{k-1} = 0$ if $k > 1$, and

$$p(z, s_0, \ldots, s_{k-1}) = p(z, \nu_k, \ldots, \nu_{2k-1}),$$

which implies $\nu_k = s_0, \ldots, \nu_{2k-1} = s_{k-1}$. It follows that $n(z)$ is a solution of Problem 3.3. □

4. Self-adjoint realization of $\widehat{n}(z)$: Cases I and II

In this and the following section we study the effect of the Schur transformation on the minimal self-adjoint realization of the Nevanlinna function $n(z) \in \mathbf{N}$. That is, we deal with the problem: How to derive from the minimal self-adjoint realization of $n(z)$ the minimal self-adjoint realization of its Schur transform $\widehat{n}(z)$. We consider three cases in accordance with the definition of the Schur transformation:

Case I: $n(z)$ is holomorphic at z_1 and $\operatorname{Im} n(z_1) \neq 0$.

Case II: $n(z)$ has a pole at z_1. Then, by Case II of the definition of the Schur transformation, $\widehat{n}(z)$ is holomorphic at z_1.

Case III: $n(z)$ is holomorphic at z_1 and $\operatorname{Im} n(z_1) = 0$.

Case I includes the situation where $n(z)$ and hence (see Remark 2.1 (3)) $\widehat{n}(z)$ are Nevanlinna functions, that is, where the underlying state spaces in the self-adjoint realizations are Hilbert spaces. Case II plays a key role in the study of Case III. In this section we deal with Cases I and II, in Section 5 we treat Case III, and in Section 6 we consider the situation where in Cases I and III the Schur transform $\widehat{n}(z)$ has a pole at z_1. Then, by a subsequent Schur transformation according to Case II, we obtain a function $\widetilde{n}(z)$ which is holomorphic at z_1, and we describe the minimal self-adjoint realization on $\widetilde{n}(z)$ using that of $n(z)$.

We first note that if $n(z) \in \mathbf{N}$ is holomorphic at z_1 and has the minimal self-adjoint realization

$$n(z) \sim \big(\mathcal{P}, A, \varphi(z)\big), \tag{4.1}$$

then, in particular,

$$n(z) = n(z_1)^* + (z - z_1^*)\big\langle\big(I + (z-z_1)(A-z)^{-1}\big)u, u\big\rangle_{\mathcal{P}}, \quad u = \varphi(z_1). \tag{4.2}$$

By expanding the formula on the righthand side in powers of $(z - z_1)$, we find that if $n(z)$ has Taylor expansion (2.1), its Taylor coefficients ν_i are given by $\nu_0 = n(z_1)$ and

$$\nu_i = \big\langle (A-z_1)^{-i+1}\big(I + (z_1 - z_1^*)(A-z_1)^{-1}\big)u, u\big\rangle_{\mathcal{P}}, \quad i = 1, 2, \ldots. \tag{4.3}$$

Moreover,

$$\langle u, u\rangle_{\mathcal{P}} = \mu \ \big(= (\nu_0 - \nu_0^*)/(z_1 - z_1^*)\big), \tag{4.4}$$

and for $z \in \rho(A) \setminus \{z_1, z_1^*\}$,

$$\big\langle (A-z)^{-1}u, u\big\rangle_{\mathcal{P}} = \frac{n(z) - \alpha(z)}{(z-z_1)(z-z_1^*)}, \quad \alpha(z) = \nu_0 + \mu(z - z_1). \tag{4.5}$$

The equality (4.5) follows directly from (4.2).

Case I. We assume that $n(z) \in \mathbf{N}$ is holomorphic at z_1 and that it has the minimal self-adjoint realization (4.1):

$$n(z) \sim \big(\mathcal{P}, A, \varphi(z)\big)\,.$$

With $\nu_0 = n(z_1)$, $\mu = \langle \varphi(z_1), \varphi(z_1)\rangle_{\mathcal{P}} = (\nu_0 - \nu_0^*)/(z_1 - z_1^*)\ (\neq 0)$, and the functions

$$\alpha(z) = \nu_0 + \mu(z - z_1), \quad \beta(z) = \nu_0^* - \mu(z - z_1)\,,$$

which are real $\big(\alpha(z)^* = \alpha(z^*),\ \beta(z)^* = \beta(z)^*\big)$, the Schur transform $\widehat{n}(z)$ was defined as (see (2.2))

$$\widehat{n}(z) = \frac{\beta(z)n(z) - |\nu_0|^2}{n(z) - \alpha(z)}\,.$$

To find a minimal self-adjoint realization $\big(\widehat{\mathcal{P}}, \widehat{A}, \widehat{\varphi}(z)\big)$ of $\widehat{n}(z)$ we start with the following lemma.

Lemma 4.1. *Assume $n(z) \in \mathbf{N}$ is holomorphic at z_1, $\operatorname{Im} n(z_1) \neq 0$, $n(z)$ is not linear and has the minimal self-adjoint realization $\big(\mathcal{P}, A, \varphi(z)\big)$. Then with $u = \varphi(z_1)$ the following statements hold:*

(i) *The space* $\operatorname{span}\{u\}$ *is a nondegenerate subspace of $\mathcal{P}$ and hence the orthogonal projection $\widehat{P}$ in $\mathcal{P}$ onto $\widehat{\mathcal{P}} = \mathcal{P} \ominus \operatorname{span}\{u\}$ is well defined.*

(ii) *The compression $\widehat{A}$ of A to $\widehat{\mathcal{P}}$:*

$$\widehat{A} = \widehat{P}A\big|_{\widehat{\mathcal{P}}} = \big\{\{f, \widehat{P}g\}\big|\{f,g\} \in A, f \in \widehat{\mathcal{P}}\big\}$$

is self-adjoint in $\widehat{\mathcal{P}}$ and $\emptyset \neq \big\{z \in \rho(A)\big|\langle (A-z)^{-1}u, u\rangle_{\mathcal{P}} \neq 0\big\} \subset \rho(\widehat{A})$.

(iii) *The resolvent of $\widehat{A}$ is given by*

$$\big(\widehat{A} - z\big)^{-1} = (A-z)^{-1} - \frac{\big\langle (A-z)^{-1}\,\cdot\,, u\big\rangle_{\mathcal{P}}}{\big\langle (A-z)^{-1}u, u\big\rangle_{\mathcal{P}}}\,(A-z)^{-1}u,\ \ z \in \rho(A) \cap \rho(\widehat{A})\,. \quad (4.6)$$

Proof. Item (i) follows from (4.4). We first prove item (iii). By (4.5), the set $\big\{z \in \rho(A)\big|\langle (A-z)^{-1}u, u\rangle_{\mathcal{P}} \neq 0\big\}$ is open and nonempty. From the definition of $\widehat{A}$ we see that for z in this set,

$$\begin{aligned}
\big(\widehat{A} - z\big)^{-1} &= \left\{\Big\{\widehat{P}(g - zf), (A-z)^{-1}(g - zf)\Big\}\Big|\ \{f,g\} \in A, f \in \widehat{\mathcal{P}}\right\} \\
&= \left\{\left\{h, (A-z)^{-1}\left(h + \frac{\langle g, u\rangle_{\mathcal{P}}}{\langle u, u\rangle_{\mathcal{P}}}u\right)\right\}\Big|\ h = \widehat{P}(g - zf), \{f,g\} \in A, f \in \widehat{\mathcal{P}}\right\} \\
&= \left\{\left\{h, (A-z)^{-1}h - \frac{\langle (A-z)^{-1}h, u\rangle_{\mathcal{P}}}{\langle (A-z)^{-1}u, u\rangle_{\mathcal{P}}}(A-z)^{-1}u\right\}\Big|\ h \in \widehat{\mathcal{P}}\right\}.
\end{aligned}$$

Here in the description of the sets we have used that $f \in \widehat{\mathcal{P}}$ and hence $\langle f, u\rangle_{\mathcal{P}} = 0$ and

$$\widehat{P}(g - zf) = g - zf - \frac{\langle g, u\rangle_{\mathcal{P}}}{\langle u, u\rangle_{\mathcal{P}}}u\,.$$

The last equality follows from the formula

$$(A-z)^{-1}h=(A-z)^{-1}\widehat{P}(g-zf)=f-\frac{\langle g,u\rangle_{\mathcal{P}}}{\langle u,u\rangle_{\mathcal{P}}}(A-z)^{-1}u\,,$$

which implies that

$$\frac{\langle g,u\rangle_{\mathcal{P}}}{\langle u,u\rangle_{\mathcal{P}}}=-\frac{\big\langle (A-z)^{-1}h,u\big\rangle_{\mathcal{P}}}{\big\langle (A-z)^{-1}u,u\big\rangle_{\mathcal{P}}}\,.$$

Item (ii) follows from item (iii). Indeed, a relation A in a Pontryagin space is self-adjoint if and only if its resolvent $R(z)=(A-z)^{-1}$ is symmetric with respect to the real axis: $R(z)^*=R(z^*)$ and satisfies the resolvent identity:

$$R(z)-R(w)=(z-w)R(z)R(w)\,;$$

the equality (4.6) implies that $\widehat{R}(z)=(\widehat{A}-z)^{-1}$ has these properties. The inclusion in item (ii) also follows from (4.6). □

Theorem 4.2. *Assume $n(z)\in\mathbf{N}$ is holomorphic at z_1, $\operatorname{Im} n(z_1)\neq 0$, $n(z)$ is not linear and has the minimal self-adjoint realization $\big(\mathcal{P},A,\varphi(z)\big)$. With*

$$u=\varphi(z_1),\quad \widehat{\mathcal{P}}=\mathcal{P}\ominus\operatorname{span}\{u\},\quad \widehat{A}=\widehat{P}A|_{\widehat{\mathcal{P}}}\,,$$

and

$$\widehat{\varphi}(z)=\frac{1}{z-z_1}\,\frac{\langle u,u\rangle_{\mathcal{P}}}{\langle (A-z)^{-1}u,u\rangle_{\mathcal{P}}}\widehat{P}\varphi(z)\,,$$

where $\widehat{P}$ is the orthogonal projection in $\mathcal{P}$ onto $\widehat{\mathcal{P}}$, $\widehat{n}(z)$ has the minimal self-adjoint realization $\big(\widehat{\mathcal{P}},\widehat{A},\widehat{\varphi}(z)\big)$. The space $\widehat{\mathcal{P}}$ is a Pontryagin space with negative index

$$\operatorname{ind}_-(\widehat{\mathcal{P}})=\begin{cases}\operatorname{ind}_-(\mathcal{P}) & \text{if } \operatorname{Im}\nu_0>0\,,\\ \operatorname{ind}_-(\mathcal{P})-1 & \text{if } \operatorname{Im}\nu_0<0\,.\end{cases}\tag{4.7}$$

Proof. The formula (4.7) holds since $\langle u,u\rangle_{\mathcal{P}}>0$ (<0) if $\operatorname{Im}\nu_0>0$ (<0). In this proof we use the notation

$$\gamma(z)=\frac{\langle u,u\rangle_{\mathcal{P}}}{\big\langle (A-z)^{-1}u,u\big\rangle_{\mathcal{P}}}\,.$$

The function $\widehat{\varphi}(z)$ can also be written as

$$\widehat{\varphi}(z)=\frac{\langle u,u\rangle_{\mathcal{P}}}{\big\langle (A-z)^{-1}u,u\big\rangle_{\mathcal{P}}}\widehat{P}(A-z)^{-1}u\,;$$

this follows fron the relation

$$\varphi(z)=u+(z-z_1)(A-z)^{-1}u$$

and the fact that $\widehat{P}u=0$. Then

$$\widehat{\varphi}(w)=\gamma(w)\widehat{P}(A-w)^{-1}u=-u+\gamma(w)(A-w)^{-1}u\,.$$

By the resolvent formula (4.6) and the resolvent identity we obtain

$$\big(I+(z-w)(\widehat{A}-z)^{-1}\big)\,\widehat{\varphi}(w)=-u+\gamma(z)(A-z)^{-1}u=\widehat{\varphi}(z)\,,$$

which shows that $\widehat{\varphi}(z)$ is an $\widehat{A}$-field. Moreover, also by (4.4), (4.5), and the equality $\beta(w^*) = \beta(w)^*$, we find

$$\begin{aligned}
(z-w^*)\left\langle \widehat{\varphi}(z), \widehat{\varphi}(w)\right\rangle_{\widehat{\mathcal{P}}} &= (z-w^*)\left\langle -u+\gamma(z)(A-z)^{-1}u, -u+\gamma(w)(A-w)^{-1}u\right\rangle_{\widehat{\mathcal{P}}} \\
&= -\mu(z-w^*) - \frac{\mu^2}{\left\langle (A-z)^{-1}u, u\right\rangle_{\mathcal{P}}} + \frac{\mu^2}{\left\langle u, (A-w)^{-1}u\right\rangle_{\mathcal{P}}} \\
&= \nu_0^* - \mu(z-z_1) - \frac{\mu^2}{\left\langle (A-z)^{-1}u, u\right\rangle_{\mathcal{P}}} \\
&\quad - \left(\nu_0^* - \mu(w^*-z_1) - \frac{\mu^2}{\left\langle u, (A-w)^{-1}u\right\rangle_{\mathcal{P}}}\right) \\
&= \beta(z) - \frac{\mu^2(z-z_1)(z-z_1^*)}{n(z)-\alpha(z)} \\
&\quad - \left(\beta(w) - \frac{\mu^2(w-z_1)(w-z_1^*)}{n(w)-\alpha(w)}\right)^* \\
&= \widehat{n}(z) - \widehat{n}(w)^*\,,
\end{aligned}$$

which completes the proof that $\big(\widehat{\mathcal{P}}, \widehat{A}, \widehat{\varphi}(z)\big)$ is a self-adjoint realization of $\widehat{n}(z)$. The minimality of the realization follows directly from the formula for $\widehat{\varphi}(z)$ and the fact that $\varphi(z)$ is minimal. □

In the theorem $\widehat{n}(z)$ may have a pole at z_1. But if $\widehat{n}(z)$ is holomorphic at z_1, then $z_1 \in \rho(\widehat{A}) \cap \rho(A)$ and, by (4.5), Remark 2.1 (2), and (2.4),

$$\left\langle (A-z_1)^{-1}u, u\right\rangle_{\mathcal{P}} = \frac{\nu_1 - \mu}{z_1 - z_1^*} \neq 0\,.$$

This readily implies the following result.

Corollary 4.3. *If in Theorem* 4.2 $\widehat{n}(z)$ *is holomorphic at* z_1*, then its minimal self-adjoint realization centered at* z_1 *is given by*

$$\widehat{n}(z) = \widehat{n}(z_1)^* + (z-z_1^*)\left\langle \big(I + (z-z_1)(\widehat{A}-z)^{-1}\big)\widehat{u}, \widehat{u}\right\rangle_{\widehat{\mathcal{P}}}, \quad z \in \rho(\widehat{A})\,,$$

where

$$\widehat{u} = \widehat{\varphi}(z_1) = \frac{\nu_0 - \nu_0^*}{\nu_1 - \mu}\widehat{P}(A-z_1)^{-1}u, \quad \widehat{n}(z_1) = \frac{\nu_0^*\nu_1 - \nu_0\mu}{\nu_1 - \mu}\,.$$

Case II. We suppose now that $n(z) \in \mathbf{N}$ has a pole of order q at z_1. Then it has also a pole of the same order at z_1^*; the corresponding principal parts we denote by $h_{z_1}(z)$ and $h_{z_1^*}(z)$. Let again $\big(\mathcal{P}, A, \varphi(z)\big)$ be a minimal self-adjoint realization $n(z)$. Then z_1 and z_1^* are eigenvalues of A, the corresponding (q-dimensional) algebraic eigenspaces we denote by $\mathcal{L}_{z_1}$ and $\mathcal{L}_{z_1^*}$. Then the space

$$\mathcal{P}_0 = \mathcal{L}_{z_1} \dotplus \mathcal{L}_{z_1^*}\,,$$

where $\dotplus$ stands for direct sum, is a $2q$-dimensional nondegenerate subspace of $\mathcal{P}$ with negative index $\operatorname{ind}_-(\mathcal{P}_0) = q$.

Theorem 4.4. *Let $n(z) \in \mathbf{N}$ have the minimal self-adjoint realization $\big(\mathcal{P}, A, \varphi(z)\big)$ and assume that $n(z)$ has poles at z_1 and z_1^* of order q. Then its Schur transform*

$$\widehat{n}(z) = n(z) - h_{z_1}(z) - h_{z_1^*}(z)$$

has the minimal self-adjoint realization $\big(\widehat{\mathcal{P}}, \widehat{A}, \widehat{\varphi}(z)\big)$, where

$$\widehat{\mathcal{P}} = \mathcal{P} \ominus \mathcal{P}_0\,,$$

$\widehat{A}$ is the restriction of A to its nondegenerate invariant subspace $\widehat{\mathcal{P}}$ which is a Pontryagin space with negative index

$$\operatorname{ind}_-\big(\widehat{\mathcal{P}}\big) = \operatorname{ind}_-(\mathcal{P}) - q\,,$$

and, if $\widehat{P}$ denotes the orthogonal projection in $\mathcal{P}$ onto $\widehat{\mathcal{P}}$,

$$\widehat{\varphi}(z) = \widehat{P}\varphi(z)\,.$$

Proof. We write $A = \widehat{A} \oplus A_0$ in accordance with the decomposition $\mathcal{P} = \widehat{\mathcal{P}} \oplus \mathcal{P}_0$, $P_0 = I - \widehat{P}$, and $\widehat{\varphi}(z) = \widehat{P}\,\varphi(z)$, $\varphi_0(z) = P_0\,\varphi(z)$. Then, with an arbitrary point $w \in \rho(A)$,

$$\begin{aligned}
n(z) &= n(w)^* + (z-w^*)\big\langle\big(I + (z-w)(A-z)^{-1}\big)\varphi(w), \varphi(w)\big\rangle_{\mathcal{P}}\\
&= n(w)^* + (z-w^*)\big\langle\big(I + (z-w)(\widehat{A}-z)^{-1}\big)\widehat{\varphi}(w), \widehat{\varphi}(w)\big\rangle_{\mathcal{P}}\\
&\quad + (z-w^*)\big\langle(A_0-z)^{-1}(A_0-w)\varphi_0(w), \varphi_0(w)\big\rangle_{\mathcal{P}}\\
&= n(w)^* + (z-w^*)\big\langle\big(I + (z-w)(\widehat{A}-z)^{-1}\big)\widehat{\varphi}(w), \widehat{\varphi}(w)\big\rangle_{\mathcal{P}}\\
&\quad - \big\langle(A_0-w)\varphi_0(w), \varphi_0(w)\big\rangle\\
&\quad + \big\langle(A_0-z)^{-1}(A_0-w)\varphi_0(w), (A_0-w)\varphi_0(w)\big\rangle_{\mathcal{P}}\,,
\end{aligned}$$

which implies

$$\big\langle(A_0-z)^{-1}(A_0-w)\varphi_0(w), (A_0-w)\varphi_0(w)\big\rangle_{\mathcal{P}} = h_{z_1}(z) + h_{z_1^*}(z)\,. \tag{4.8}$$

From the relation

$$\begin{aligned}
\big(I + (z-w)(\widehat{A}-z)^{-1}\big)\widehat{\varphi}(w) &= \big(I + (z-w)(\widehat{A}-z)^{-1}\big)\widehat{P}\varphi(w)\\
&= \widehat{P}\,\big(I + (z-w)(A-z)^{-1}\big)\varphi(w) = \widehat{P}\,\varphi(z) = \widehat{\varphi}(z)
\end{aligned}$$

it follows that $\widehat{\varphi}(z)$ is a $\widehat{A}$-field. Using again that

$$(z-w^*)\big(I + (z-w)(A_0-z)^{-1}\big) = (A_0-w^*)(A_0-z)^{-1}(A_0-w) - (A_0-w)$$

we find with (4.8)

$$\begin{aligned}
&(z-w^*)\big\langle\widehat{\varphi}(z), \widehat{\varphi}(w)\big\rangle_{\widehat{\mathcal{P}}}\\
&\qquad = (z-w^*)\big\langle\varphi(z), \varphi(w)\big\rangle_{\mathcal{P}} - (z-w^*)\big\langle P_0\varphi(z), P_0\varphi(w)\big\rangle_{\mathcal{P}}\\
&\qquad = n(z) - n(w)^* - (z-w^*)\big\langle\big(I + (z-w)(A_0-z)^{-1}\big)\varphi_0(w), \varphi_0(w)\big\rangle_{\mathcal{P}}
\end{aligned}$$

$$\begin{aligned}&= n(z) - \big\langle (A_0 - z)^{-1}(A_0 - w)\varphi_0(w), (A_0 - w)\varphi_0(w)\big\rangle_{\mathcal{P}}\\&\quad - \Big(n(w) - \big\langle (A_0 - w)^{-1}(A_0 - w)\varphi_0(w), (A_0 - w)\varphi_0(w)\big\rangle_{\mathcal{P}}\Big)^*\\&= \widehat{n}(z) - \widehat{n}(w)^* .\end{aligned}$$

□

Since $\widehat{n}(z)$ in the theorem is holomorphic at z_1, we have the following result in which z_0 is an arbitrary point in $\rho(A)$; in particular, $z_0 \neq z_1$.

Corollary 4.5. *The function $\widehat{n}(z)$ in Theorem 4.4 has the minimal self-adjoint realization centered at z_1 given by*

$$\widehat{n}(z) = \widehat{n}(z_1)^* + (z - z_1^*)\big\langle \big(I + (z - z_1)(\widehat{A} - z)^{-1}\big)\widehat{u}, \widehat{u}\big\rangle_{\widehat{\mathcal{P}}} ,$$

where

$$\widehat{u} = \widehat{\varphi}(z_1) = \big(I + (z_1 - z_0)(\widehat{A} - z_1)^{-1}\big)\widehat{P}\varphi(z_0)$$

and

$$\widehat{n}(z_1) = \lim_{z \to z_1} \big(n(z) - h_{z_1}(z)\big) - h_{z_1^*}(z_1) .$$

5. Self-adjoint realization of $\widehat{n}(z)$: case III

In **Case III** we assume that $n(z) \in \mathbf{N}$ has Taylor expansion (2.1) at z_1 with $\operatorname{Im}\nu_0 = \operatorname{Im} n(z_1) = 0$ and the minimal self-adjoint realization (4.1):

$$n(z) \sim \big(\mathcal{P}, A, \varphi(z)\big) .$$

Throughout this section we assume that $-1/(n(z) - \nu_0)$ is not a rational function with poles only in z_1, z_1^* and vanishing at ∞. Then the Schur transform $\widehat{n}(z)$ is not identically equal to ∞. We will use the following result (compare with [11, Section 3] and [6, Theorem 3.3 and Subsection 7.2]).

Lemma 5.1. *Assume $f(z) \in \mathbf{N}_\kappa$ is not identically equal to a real number and has the minimal self-adjoint realization*

$$f(z) \sim \big(\mathcal{P}, A, \varphi(z)\big) .$$

Then the following statements hold.

(i) *The function $f_i(z) = -1/f(z)$ belongs to the class $\mathbf{N}_\kappa$ and has the minimal self-adjoint realization*

$$f_i(z) \sim \big(\mathcal{P}, A_i, \varphi_i(z)\big) ,$$

where A_i is the self-adjoint relation in $\mathcal{P}$ with resolvent given by

$$(A_i - z)^{-1} = (A - z)^{-1} - \frac{\big\langle \,\cdot\,, \varphi(z^*)\big\rangle_{\mathcal{P}}}{f(z)}\,\varphi(z) , \tag{5.1}$$

and the A_i-field $\varphi_i(z)$ is given by

$$\varphi_i(z) = \frac{\varphi(z)}{f(z)} .$$

(ii) *Suppose that $\lambda \in \rho(A)$. Then λ is an eigenvalue of A_i and the corresponding root space $\mathcal{L}_\lambda(A_i)$ has dimension m if and only if λ is a zero of order $m \geq 1$ of $f(z)$. In this case the elements*

$$\varphi(\lambda),\ \frac{\varphi^{(1)}(\lambda)}{1!},\ \ldots,\ \frac{\varphi^{(m-1)}(\lambda)}{(m-1)!} \tag{5.2}$$

form a Jordan chain of A_i at λ with eigenvector $\varphi(\lambda)$.

Proof. (i) For $z \in \operatorname{hol}(f) \cap \operatorname{hol}(f_i)$, the expression on the righthand side of formula (5.1) is symmetric with respect to the real axis and satisfies the resolvent identity (see the last part of the proof of Lemma 4.1) and hence is the resolvent of a self-adjoint linear relation in $\mathcal{P}$ which we denote by A_i. Consider the $\mathcal{P}$-valued function

$$\varphi_i(z) := \big(I + (z-z_0)(A_i - z)^{-1}\big)\frac{\varphi(z_0)}{f(z_0)}\,. \tag{5.3}$$

With resolvent identity one can show that it is an A_i-field. Replacing the resolvent in (5.3) by the righthand side of formula (5.1) we obtain

$$\begin{aligned}\varphi_i(z) &= \big(I + (z-z_0)(A-z)^{-1}\big)\frac{\varphi(z_0)}{f(z_0)} - (z-z_0)\frac{\big\langle \varphi(z_0), \varphi(z^*)\big\rangle}{f(z)f(z_0)}\,\varphi(z)\\ &= \frac{\varphi(z)}{f(z_0)} + \frac{f(z_0)-f(z)}{f(z)f(z_0)}\,\varphi(z) \ = \frac{\varphi(z)}{f(z)}\,.\end{aligned}$$

It follows that

$$\begin{aligned}\big\langle \varphi_i(z), \varphi_i(w)\big\rangle_{\mathcal{P}} &= \frac{\big\langle \varphi(z), \varphi(w)\big\rangle_{\mathcal{P}}}{f(z)f(w)^*} = \frac{f(z)-f(w)^*}{(z-w^*)f(z)f(w)^*}\\ &= \frac{1}{z-w^*}\left(-\frac{1}{f(z)} + \frac{1}{f(w)^*}\right) = L_{f_i}(z,w)\,.\end{aligned}$$

Hence the triple $(\mathcal{P}, A_i, \varphi_i(z))$ is a self-adjoint realization of $f_i(z)$. That it is minimal follows from the relation $\varphi_i(z) = \varphi(z)/f(z)$ and the minimality of $\big(\mathcal{P}, A, \varphi(z)\big)$.

(ii) In the proof of (ii) we can suppose that A_i is an operator (otherwise we restrict the considerations to the subspace generated by the span of the root subspaces of A_i at λ and λ^*). We use the following fact: If the resolvent of the linear operator T has a pole at λ with Laurent expansion

$$(T-z)^{-1} = \sum_{j=1}^{m} \frac{D_j}{(z-\lambda)^j} + \cdots\,, \tag{5.4}$$

where the terms in $\cdots$ are holomorphic at λ, then

$$(T-\lambda)D_j = D_{j+1}, \quad j = 1, \ldots, m-1, \quad (T-\lambda)D_m = 0 \tag{5.5}$$

(compare with [9, I.5.10]). For $T = A_i$ in (5.4) we obtain from (5.1) with $f(z) = (z-\lambda)^m\, g(z)$, $g(\lambda) \neq 0$, and $\psi(z) = \varphi(z)/g(z)$

$$D_j = \sum_{\mu=0}^{m-j} \left\langle\ \cdot\ , \frac{\varphi^{(\mu)}(\lambda^*)}{\mu!}\right\rangle_{\mathcal{P}} \frac{\psi^{(m-j-\mu)}(\lambda)}{(m-j-\mu)!}, \quad j = 1, 2, \ldots, m\,.$$

Now the last relation in (5.5) yields for $x \in \mathcal{P}$,

$$(A_i - \lambda)\big\langle x, \varphi(\lambda^*)\big\rangle_{\mathcal{P}}\psi(\lambda) = 0\,.$$

Substituting $x = \varphi(z)/(z-\lambda)^m$ and letting $z \to \lambda$ we find

$$(A_i - \lambda)\varphi(\lambda) = 0\,. \tag{5.6}$$

From the first relation in (5.5) for $m-1$ we obtain

$$(A_i - \lambda)\left(\left\langle\, x, \frac{\varphi^{(1)}(\lambda^*)}{1!}\right\rangle_{\mathcal{P}} \psi(\lambda) + \big\langle\, x, \varphi(\lambda^*)\big\rangle_{\mathcal{P}}\, \frac{\psi^{(1)}(\lambda)}{1!}\right) = \big\langle\, x, \varphi(\lambda^*)\big\rangle_{\mathcal{P}}\, \psi(\lambda)$$

which, with the same choice of x and because of (5.6), yields

$$(A_i - \lambda)\,\frac{\varphi^{(1)}(\lambda)}{1!} = \varphi(\lambda)\,.$$

This reasoning can be continued to yield (5.2). Conversely, if $f(\lambda) \neq 0$ then according to (5.1) the resolvent of A_i is holomorphic at λ and hence $\lambda \in \rho(A_i)$. It follows from the form of D_j that there is only one chain, so the geometric simplicity of the eigenvalue follows from this argument. □

Theorem 5.2. *Assume that $n(z) \in \mathbf{N}$ is holomorphic at z_1 with Taylor expansion* (2.1) *and* $\operatorname{Im} n(z_1) = 0$, *that it is not a real constant, and that $-1/(n(z)-\nu_0)$ is not a rational function with poles only in z_1, z_1^* and vanishing at ∞, and let $\big(\mathcal{P}, A, \varphi(z)\big)$ be a minimal self-adjoint realization of $n(z)$. If k denotes the smallest integer ≥ 1 such that $\nu_k \neq 0$, we set with $u = \varphi(z_1)$:*

$$\mathcal{L} = \operatorname{span}\Big\{u, (A-z_1)^{-1}u, \ldots, (A-z_1)^{-k+1}u,$$
$$(A-z_1^*)^{-1}u, (A-z_1^*)^{-2}u, \ldots, (A-z_1^*)^{-k}u\Big\}.$$

Then $\widehat{n}(z)$ has the minimal self-adjoint realization $\big(\widehat{\mathcal{P}}, \widehat{A}, \widehat{\varphi}(z)\big)$ with

$$\widehat{\mathcal{P}} = \mathcal{P} \ominus \mathcal{L}, \quad \widehat{A} = \widehat{P}A\,|_{\widehat{\mathcal{P}}}\,,$$

and

$$\widehat{\varphi}(z) = \frac{(z-z_1)^k(z-z_1^*)^k}{p(z)\big(n(z)-\alpha(z)\big)}\,\widehat{P}\varphi(z)\,, \tag{5.7}$$

where $\widehat{P}$ is the orthogonal projection in $\mathcal{P}$ onto $\widehat{\mathcal{P}}$. The space $\widehat{\mathcal{P}}$ is a Pontryagin space with negative index

$$\operatorname{ind}_-\big(\widehat{\mathcal{P}}\big) = \operatorname{ind}_-(\mathcal{P}) - k\,.$$

The function $\widehat{\varphi}(z)$ can also be written as

$$\widehat{\varphi}(z) = \frac{(z-z_1)^k(z-z_1^*)^k}{p(z)\big(n(z)-\alpha(z)\big)}\,\widehat{P}(A-z)^{-1}(A-z_1)^{-k+1}u\,;$$

this follows from expanding $\varphi(z) = u + (z - z_1)(A - z)^{-1}u$ around z_1:

$$\varphi(z) = \sum_{j=0}^{k-1}(z - z_1)^j (A - z_1)^{-j} u + (z - z_1)^k (A - z)^{-1}(A - z_1)^{-k+1} u \tag{5.8}$$

and using that $\widehat{P}(A - z_1)^{-j}u = 0$, $j = 0, 1, \dots k - 1$. For the scalar factor on the righthand side of (5.7), from (2.9) we obtain

$$\frac{(z - z_1)^k (z - z_1^*)^k}{p(z)\big(n(z) - \alpha(z)\big)} = \frac{\widehat{n}(z) - \nu_0}{n(z) - \nu_0} = \frac{\beta(z) - \nu_0}{n(z) - \alpha(z)}\,. \tag{5.9}$$

In the corollary below we suppose that $\widehat{n}(z)$ is holomorphic at z_1. Then also $\widehat{\varphi}(z)$ is holomorphic at z_1. Taking limits as $z \to z_1$ and using Remark 2.3 (1) we find that

$$\widehat{\varphi}(z_1) = \frac{(z_1 - z_1^*)^k}{\nu_k(b_k - a_k)}\,\widehat{P}(A - z_1)^{-k}u\,.$$

This implies the following result.

Corollary 5.3. *If in Theorem* 5.2 $\widehat{n}(z)$ *is holomorphic at* z_1, *then it has the minimal self-adjoint realization centered at* z_1 *given by*

$$\widehat{n}(z) = \widehat{n}(z_1)^* + (z - z_1^*)\Big\langle\big(I + (z - z_1)(\widehat{A} - z)^{-1}\big)\widehat{u}, \widehat{u}\Big\rangle_{\widehat{\mathcal{P}}}\,,$$

where

$$\widehat{n}(z_1) = \nu_0 - \frac{(z_1 - z_1^*)^k}{b_k - a_k}, \quad \widehat{u} = \frac{(z_1 - z_1^*)^k}{\nu_k(b_k - a_k)}\,\widehat{P}(A - z_1)^{-k}u\,.$$

Proof of Theorem 5.2. The Schur transform of $n(z)$ can be written as

$$\widehat{n}(z) - \nu_0 = \frac{-1}{\dfrac{-1}{n(z) - \nu_0} - \dfrac{-p(z)}{(z - z_1)^k (z - z_1^*)^k}} = \Big[\big[[n(z) - \nu_0]_i\big]^{\wedge}\Big]_i\,, \tag{5.10}$$

that is, on account of (2.5), as the negative reciprocal of the Schur transform of the negative reciprocal of $n(z) - \nu_0$. Consequently, to obtain the desired realization we first apply Lemma 5.1 to $n(z) - \nu_0$, then the Schur transformation Case II, which amounts to removing the poles at z_1 and z_1^* from $[n(z) - \nu_0]_i$, and then again Lemma 5.1.

By Lemma 5.1, we have

$$[n(z) - \nu_0]_i \sim \big(\mathcal{P}, A_i, \varphi_i(z)\big)\,,$$

where A_i is the self-adjoint relation with resolvent

$$(A_i - z)^{-1} = (A - z)^{-1} - \frac{\big\langle\,\cdot\,, \varphi(z^*)\big\rangle_{\mathcal{P}}}{n(z) - \nu_0}\,\varphi(z) \tag{5.11}$$

and $\varphi_i(z) = \varphi(z)/(n(z) - \nu_0)$. Then by definition of the Schur transformation

$$\big[\,[n(z) - \nu_0]_i\,\big]^{\wedge} \sim \big(\widehat{\mathcal{P}}, \widehat{A_i}, \widehat{\varphi}_i(z)\big)\,. \tag{5.12}$$

Here $\widehat{P}$ is as in the theorem. Indeed, according to Lemma 5.1 (ii), $\mathcal{L}_{z_1}$ and $\mathcal{L}_{z_1^*}$, where

$$\mathcal{L}_w = \operatorname{span}\left\{\varphi(w), \frac{1}{1!}\varphi'(w), \ldots, \frac{1}{(k-1)!}\varphi^{(k-1)}(w)\right\}, \quad w \in \{z_1, z_1^*\},$$

are the root spaces of A_i for the eigenvalues z_1, and z_1^* and by calculating the derivatives of $\varphi(z)$ we find that

$$\mathcal{L} = \mathcal{L}_{z_1} \dotplus \mathcal{L}_{z_1^*}.$$

It follows that $\mathcal{L}$ is a $2k$-dimensional Pontryagin subspace with $\operatorname{ind}_-(\mathcal{L}) = k$, which implies the formula for the negative index of $\widehat{\mathcal{P}}$. Since $\mathcal{L}$ is A_i-invariant and A_i is self-adjoint, $\widehat{\mathcal{P}}$ is also A_i-invariant. Returning to (5.12) we have that $\widehat{A_i}$ is the restriction of A_i to $\widehat{\mathcal{P}}$ and that

$$\widehat{\varphi}_i(z) = \widehat{P}\varphi_i(z) = \widehat{P}\varphi(z)/(n(z) - \nu_0), \tag{5.13}$$

where $\widehat{P}$ is the orthogonal projection in $\mathcal{P}$ onto $\widehat{P}$. Finally, by applying Lemma 5.1 again, we obtain the minimal self-adjoint realization

$$\widehat{n}(z) - \nu_0 \sim \big(\widehat{\mathcal{P}}, \widehat{A}, \widehat{\varphi}(z)\big),$$

with the self-adjoint relation $\widehat{A}$ given by its resolvent

$$\big(\widehat{A} - z\big)^{-1} = \big(\widehat{A_i} - z\big)^{-1} - \big(\widehat{n}(z) - \nu_0\big)\left\langle \,\cdot\,, \widehat{\varphi}_i(z^*)\right\rangle_{\widehat{\mathcal{P}}} \widehat{\varphi}_i(z) \tag{5.14}$$

and

$$\widehat{\varphi}(z) = \frac{\widehat{n}(z) - \nu_0}{n(z) - \nu_0}\, \widehat{P}\varphi(z).$$

It remains to prove that $\widehat{A}$ is the compression of A to $\widehat{\mathcal{P}}$. To this end, we rewrite (5.11) as

$$(A - z)^{-1} = (A_i - z)^{-1} - \big(n(z) - \nu_0\big)\left\langle \,\cdot\,, \varphi_i(z^*)\right\rangle_{\mathcal{P}} \varphi_i(z),$$

which implies that

$$A = \Big\{\{f, g\} + c\{\varphi_i(z_0), z_0\varphi_i(z_0)\}\big|\{f, g\} \in A_i,$$
$$c = \big(n(z_0) - \nu_0\big)\big\langle g - z_0^* f, \varphi_i(z_0)\big\rangle_{\mathcal{P}}\Big\}.$$

Now we take the compression of both sides to the subspace $\widehat{\mathcal{P}}$ and we claim that

$$\widehat{P}A\big|_{\widehat{\mathcal{P}}} = \Big\{\{x, y\} + c\{\widehat{\varphi}_i(z_0), z_0\widehat{\varphi}_i(z_0)\}\big|\{x, y\} \in \widehat{A}_i,$$
$$c = (\widehat{n}(z_0) - \nu_0)\left\langle y - z_0^* x, \widehat{\varphi}_i(z_0)\right\rangle_{\mathcal{P}}\Big\}.$$

To see this we note that

$$f + c\varphi_i(z_0) \in \widehat{\mathcal{P}} \iff f = \widehat{P}f - c(I - \widehat{P})\varphi_i(z_0),$$

and in this case,

$$\widehat{P}A\big|_{\widehat{\mathcal{P}}} \ni \Big\{f + c\varphi_i(z_0), \widehat{P}\big(g + z_0\varphi_i(z_0)\big)\Big\} = \{x, y\} + c\big\{\widehat{\varphi}_i(z_0), z_0\widehat{\varphi}_i(z_0)\big\}$$

with $\{x,y\} = \{\widehat{P}f, \widehat{P}g\} \in \widehat{A_i}$ and

$$\begin{aligned}
\frac{c}{n(z_0)-\nu_0} &= \langle g - z_0^* f, \varphi_i(z_0)\rangle_{\mathcal{P}} \\
&= \langle \widehat{P}(g - z_0^* f), \widehat{P}\varphi_i(z_0)\rangle_{\mathcal{P}} + \langle (I-\widehat{P})(g - z_0^* f), (I-\widehat{P})\varphi_i(z_0)\rangle_{\mathcal{P}} \\
&\overset{1}{=} \langle y - z_0^* x, \widehat{\varphi}_i(z_0)\rangle_{\mathcal{P}} + \langle (A_{i0} - z_0^*)(I-\widehat{P})f, (I-\widehat{P})\varphi_i(z_0)\rangle_{\mathcal{P}} \\
&= \langle y - z_0^* x, \widehat{\varphi}_i(z_0)\rangle_{\mathcal{P}} - c\big\langle (I-\widehat{P})\varphi_i(z_0), (A_{i0} - z_0)(I-\widehat{P})\varphi_i(z_0)\big\rangle \\
&\overset{2}{=} \langle y - z_0^* x, \widehat{\varphi}_i(z_0)\rangle + \frac{c\,p(z_0)}{(z_0 - z_1)^k (z_0 - z_1^*)^k},
\end{aligned}$$

where on the right of $\overset{1}{=}$ the operator A_{i0} is the restriction of A_i to its invariant subspace $\mathcal{L}$ and the equality $\overset{2}{=}$ is the analog of (4.8) for the generalized Nevanlinna function $-1/(n(z)-\nu_0)$. It follows that

$$\langle y - z_0^* x, \widehat{\varphi}_i(z_0)\rangle = \frac{c}{n(z_0)-\nu_0} - \frac{c\,p(z_0)}{(z_0 - z_1)^k (z_0 - z_1^*)^k} = \frac{c}{\widehat{n}(z)-\nu_0},$$

which completes the proof of the claim. Now we use (5.14) to obtain

$$\widehat{A} = \Big\{ \{x,y\} + c\{\widehat{\varphi}_i(z_0), z_0\widehat{\varphi}_i(z_0)\} \,\big|\, \{x,y\} \in \widehat{A_i},\ c = \big(\widehat{n}(z_0) - \nu_0\big)\langle y - z_0^* x, \widehat{\varphi}_i(z_0)\rangle_{\mathcal{P}} \Big\},$$

hence $\widehat{A} = \widehat{P}A\,|_{\widehat{\mathcal{P}}}$. □

In the remainder of this section we show that in the situation of Theorem 5.2 a representation of the form as in (4.6) for the resolvent of the compression $\widehat{A}$ holds. We use the following vector notation. If

$$\mathfrak{B} = \begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix}$$

is a row vector whose entries b_j are elements of the Pontryagin space $\mathcal{P}$ and B is a bounded operator in $\mathcal{P}$ we define

$$B\mathfrak{B} = \begin{pmatrix} Bb_1 & Bb_2 & \dots & Bb_m \end{pmatrix}.$$

If $\mathfrak{C} = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}$ is a second row vector of elements from $\mathcal{P}$ then $\langle \mathfrak{C}, \mathfrak{B}\rangle_{\mathcal{P}}$ is the $m \times n$-matrix

$$\langle \mathfrak{C}, \mathfrak{B}\rangle_{\mathcal{P}} = (\gamma_{ij})_{i=1,\dots,m, j=1,\dots,n} \quad \text{with } \gamma_{ij} = \langle c_j, b_i\rangle_{\mathcal{P}}.$$

In particular, $\langle \mathfrak{B}, \mathfrak{B}\rangle_{\mathcal{P}}$ is the Gram matrix of the entries of $\mathfrak{B}$. Further, if E and F are matrices with complex entries and m and n rows, respectively, then the relation

$$\langle \mathfrak{C}F, \mathfrak{B}E\rangle_{\mathcal{P}} = E^* \langle \mathfrak{C}, \mathfrak{B}\rangle_{\mathcal{P}} F$$

holds. Further, the linear span of the entries of $\mathfrak{B}$ is denoted by $\operatorname{span}\mathfrak{B}$.

Theorem 5.4. *Let A be a self-adjoint relation in a Pontryagin space $\mathcal{P}$ with nonempty resolvent set $\rho(A)$. Let $\mathfrak{B}$ be a row vector with entries which are linearly independent elements from $\mathcal{P}$ and span a nondegenerate subspace of $\mathcal{P}$. Denote by*

P_1 the orthogonal projection in $\mathcal{P}$ onto $\mathcal{P}_1 = \mathcal{P} \ominus \operatorname{span} \mathfrak{B}$. Then the compression of A to $\mathcal{P}_1$:

$$P_1 A|_{\mathcal{P}_1} = \big\{\{f, P_1 g\} \big| \{f, g\} \in A, f \in \mathcal{P}_1\big\}$$

is a self-adjoint relation, and if

$$\det \big\langle (A-z)^{-1}\mathfrak{B}, \mathfrak{B}\big\rangle_{\mathcal{P}} \not\equiv 0$$

then $\rho(A_1)$ is not empty and the resolvent of A_1 is given by

$$(A_1 - z)^{-1}h = (A{-}z)^{-1}h - (A{-}z)^{-1}\mathcal{B}\big\langle (A{-}z)^{-1}\mathfrak{B}, \mathfrak{B}\big\rangle_{\mathcal{P}}^{-1}\big\langle (A{-}z)^{-1}h, \mathcal{B}\big\rangle_{\mathcal{P}}, \quad h \in \mathcal{P}\,. \tag{5.15}$$

If $\mathfrak{B} = (u)$ then (5.15) coincides with (4.6). The proof of the theorem is a vector version of the proof of this formula.

Proof of Theorem 5.4. That the compression is self-adjoint in the Hilbert space case due to Stenger [12] and generalized to the Pontryagin space setting in [7]. By assumption, the open set $\Omega = \big\{z \in \rho(A) \big| \det \big\langle (A-z)^{-1}\mathfrak{B}, \mathfrak{B}\big\rangle_{\mathcal{P}} \neq 0\big\}$ is nonempty. From the definition of A_1 we see that for z in this set,

$$\begin{aligned}\big(A_1 - z\big)^{-1} &= \Big\{\{P_1(g - zf), (A-z)^{-1}(g - zf)\}\,\big|\, \{f, g\} \in A, f \in \mathcal{P}_1\Big\} \\ &= \Big\{\{h, (A{-}z)^{-1}(h{+}\mathfrak{B}\langle \mathfrak{B}, \mathfrak{B}\rangle_{\mathcal{P}}^{-1}\langle g, \mathfrak{B}\rangle_{\mathcal{P}}\}\,\big|\, h{=}P_1(g{-}zf), \\ &\qquad \{f, g\}{\in} A, f{\in}\mathcal{P}_1\Big\} \\ &\stackrel{1}{=} \Big\{\{h, (A{-}z)^{-1}h{-}(A{-}z)^{-1}\mathfrak{B}\big\langle (A{-}z)^{-1}\mathfrak{B}, \mathfrak{B}\big\rangle_{\mathcal{P}}^{-1}\big\langle (A{-}z)^{-1}h, \mathcal{B}\big\rangle_{\mathcal{P}}\}\,\big| \\ &\qquad h{\in}\widehat{\mathcal{P}}\Big\}\,.\end{aligned}$$

Here we have used that if $f \in \mathcal{P}_1$, then $\langle f, \mathfrak{B}\rangle_{\mathcal{P}} = 0$ and

$$P_1(g - zf) = g - zf - \mathfrak{B}\langle \mathfrak{B}, \mathfrak{B}\rangle_{\mathcal{P}}^{-1}\langle g, \mathfrak{B}\rangle_{\mathcal{P}}\,.$$

The equality $\stackrel{1}{=}$ follows from the formula

$$(A-z)^{-1}h = (A-z)^{-1}P_1(g - zf) = f - (A-z)^{-1}\mathfrak{B}\langle \mathfrak{B}, \mathfrak{B}\rangle_{\mathcal{P}}^{-1}\langle g, \mathfrak{B}\rangle_{\mathcal{P}}$$

which implies that

$$\langle \mathfrak{B}, \mathfrak{B}\rangle_{\mathcal{P}}^{-1}\langle g, \mathfrak{B}\rangle_{\mathcal{P}} = -\big\langle (A-z)^{-1}\mathfrak{B}, \mathfrak{B}\big\rangle_{\mathcal{P}}^{-1}\big\langle (A-z)^{-1}h, \mathfrak{B}\big\rangle_{\mathcal{P}}\,.$$

The set on the righthand side of $\stackrel{1}{=}$ is the graph of a bounded operator, hence $(A_1 - z)^{-1}$ is a bounded operator for all z in Ω. Consequently, $\Omega \subset \rho(A_1)$, in particular, $\rho(A_1) \neq \emptyset$. □

Lemma 5.5. *Assume A is a self-adjoint relation in a Pontryagin space $\mathcal{P}$ with nonempty resolvent set $\rho(A)$ and let $z_1 \in \rho(A) \cap \mathbb{C}^+$. With $u \in \mathcal{P}$ let $\mathfrak{B}$ be the row vector*

$$\mathfrak{B} = \big(u \;\; (A-z_1)^{-1}u \;\; \cdots \;\; (A-z_1)^{-\ell}u \;\; (A-z_1^*)^{-1}u \;\; \cdots \;\; (A-z_1^*)^{-m}u\big)$$

and assume that the subspace span $\mathfrak{B}$ *of* $\mathcal{P}$ *is nondegenerate. If*

$$\det\left\langle (A-z)^{-1}\mathfrak{B},\mathfrak{B}\right\rangle_{\mathcal{P}} \equiv 0\,,$$

then the generalized Nevanlinna function

$$n(z)=n(z_1)^*+(z-z_1^*)\left\langle\big(I+(z-z_1)(A-z)^{-1}\big)u,u\right\rangle_{\mathcal{P}}$$

is rational and has McMillan degree at most $\ell+m+1$*.*

Proof. With $R(z)=(A-z)^{-1}$ the identities

$$R(z)R^j(w)=\frac{R(z)R(w)^{j-1}-R^j(w)}{z-w},\quad R(z)R(w)^j=\frac{R(z)}{(z-w)^j}-\sum_{s=0}^{j-1}\frac{R(w)^{j-s}}{(z-w)^{s+1}}$$

hold. Using them for $w=z_1$ and $w=z_1^*$ and elementary row operations to calculate determinants we find

$$\det\left\langle (A-z)^{-1}\mathfrak{B},\mathfrak{B}\right\rangle_{\mathcal{P}}=\frac{(-1)^{\ell+m}}{(z-z_1)^\ell(z-z_1^*)^m}\big(\langle R(z)u,u\rangle_{\mathcal{P}}\det X(z)-\det Y(z)\big)\,,$$

where $X(z)$ and $Y(z)$ are the $(\ell+m+1)\times(\ell+m+1)$ matrix functions obtained from the Gram matrix $\langle\mathfrak{B},\mathfrak{B}\rangle_{\mathcal{P}}$ by replacing its first column by

$$\begin{pmatrix} 1\\ \dfrac{1}{z-z_1^*}\\ \vdots\\ \dfrac{1}{(z-z_1^*)^\ell}\\ \dfrac{1}{z-z_1}\\ \vdots\\ \dfrac{1}{(z-z_1)^m}\end{pmatrix}\quad\text{and}\quad\begin{pmatrix} 0\\ \dfrac{\langle u,R(z_1)u\rangle_{\mathcal{P}}}{z-z_1^*}\\ \vdots\\ \sum_{s=0}^{\ell-1}\dfrac{\langle u,R(z_1)^{\ell-s}u\rangle_{\mathcal{P}}}{(z-z_1^*)^{s+1}}\\ \dfrac{\langle R(z_1)u,u\rangle_{\mathcal{P}}}{z-z_1}\\ \vdots\\ \sum_{s=0}^{m-1}\dfrac{\langle R(z_1)^{m-s}u,u\rangle_{\mathcal{P}}}{(z-z_1)^{s+1}}\end{pmatrix},$$

respectively. If $\det X(z)\equiv 0$, then the algebraic complements of the entries of the first column are equal to zero and hence $\det\langle\mathfrak{B},\mathfrak{B}\rangle_{\mathcal{P}}=0$, which contradicts the assumption of the nondegeneracy of span $\mathfrak{B}$. We find that

$$\det X(z)=\frac{q(z)}{(z-z_1^*)^\ell(z-z_1)^m},\quad \det Y(z)=\frac{r(z)}{(z-z_1^*)^\ell(z-z_1)^m}\,,$$

where $q(z)\not\equiv 0$ is a polynomial of degree at most $\ell+m$ and $r(z)$ is a polynomial of degree at most $\ell+m-1$. Since $\det\left\langle (A-z)^{-1}\mathfrak{B},\mathfrak{B}\right\rangle_{\mathcal{P}}\equiv 0$, it follows that

$$\left\langle (A-z)^{-1}u,u\right\rangle_{\mathcal{P}}=\frac{r(z)}{q(z)}\,,$$

and hence the function $n(z)$ in the theorem is a rational function of McMillan degree at most $\ell+m+1$. □

Corollary 5.6. *If, in the situation of Theorem* 5.2, *the function* $n(z)$ *is not rational of McMillan degree* $2k$, *then the resolvent of the operator* $\widehat{A}$ *in the minimal self-adjoint realization of the Schur transform* $\widehat{n}(z)$ *admits the representation*

$$(\widehat{A}-z)^{-1}=(A-z)^{-1}-(A-z)^{-1}\mathcal{B}\left\langle (A-z)^{-1}\mathcal{B},\mathcal{B}\right\rangle_{\mathcal{P}}^{-1}\left\langle (A-z)^{-1}\,\cdot\,,\mathcal{B}\right\rangle_{\mathcal{P}}$$

with

$$\mathcal{B}=\begin{pmatrix}u & (A-z_1)^{-1}u & \cdots & (A-z_1)^{-k+1}u & (A-z_1^*)^{-1}u & \cdots & (A-z_1^*)^{-k}u\end{pmatrix}.$$

The assumption about the McMillan degree in Corollary 5.6 is no real restriction since, if this assumption is not satisfied, then $\widehat{n}(z)$ is a constant (the space $\mathcal{P}$ is trivial).

6. Self-adjoint realization of the composite Schur transform

Let $n(z)$ be a generalized Nevanlinna function with Taylor expansion (2.1) at z_1. We assume that its Schur transform $\widehat{n}(z)$ is defined, $\not\equiv\infty$, and has a pole at z_1. Then the Schur transform of $\widehat{n}(z)$ is defined and *holomorphic* at z_1; we denote it by $\widetilde{n}(z)=[\,\widehat{n}\,]^{\wedge}(z)$ and call it the *composite Schur transform*. If by the Schur transformation one wants to stay within the class of holomorphic at z_1 functions, $\widetilde{n}(z)$ would be the candidate to replace $\widehat{n}(z)$.

In this section we express the minimal self-adjoint realization of $\widetilde{n}(z)$:

$$\widetilde{n}(z)\sim\big(\widetilde{\mathcal{P}},\widetilde{A},\widetilde{\varphi}(z)\big)$$

in terms of the minimal self-adjoint realization (4.1) of $n(z)$:

$$n(z)\sim\big(\mathcal{P},A,\varphi(z)\big).$$

Again we consider the two cases $\operatorname{Im}\nu_0\neq 0$ and $\operatorname{Im}\nu_0=0$ separately. The minimal self-adjoint realization $(\widehat{\mathcal{P}},\widehat{A},\widehat{\varphi}(z))$ of $\widehat{n}(z)$ in these cases is given by Theorem 4.2 and Theorem 5.2, respectively.

Case $\operatorname{Im}\nu_0\neq 0$. Recall that $\widehat{n}(z)$ has a pole of order q if and only if

$$\nu_1=\frac{\nu_0-\nu_0^*}{z_1-z_1^*},\ \nu_2=\nu_3=\cdots=\nu_q=0,\ \nu_{q+1}\neq 0\,. \tag{6.1}$$

If $q=1$ then the chain of equalities in the middle has no meaning and should be discarded. We first prove two lemmas.

Lemma 6.1. *Assume* $\operatorname{Im}\nu_0\neq 0$ *and that* (6.1) *holds. With* $u=\varphi(z_1)$ *the Gram matrix of the* $2q+1$ *elements*

$$u,(A-z_1)^{-1}u,\ldots,(A-z_1)^{-q}u,(A-z_1^*)^{-1}u,\ldots,(A-z_1^*)^{-q}u$$

is the $(2q+1)\times(2q+1)$ *matrix*

$$\widetilde{G}=\begin{pmatrix}\nu_1 & 0 & 0\\ 0 & 0 & G\\ 0 & G^* & 0\end{pmatrix},$$

where $G = \big(g_{ij}\big)_{i,j=1}^{q}$ *with*

$$g_{ij} = \big\langle (A - z_1^*)^{-j} u, (A - z_1)^{-i} u \big\rangle_{\mathcal{P}} = \big\langle (A - z_1^*)^{-i-j} u, u \big\rangle_{\mathcal{P}}$$

is an invertible $q \times q$ *Hankel matrix which is triangular with zeros above the second main diagonal* $i + j = q + 1$ *and*

$$g_{i,q+1-i} = -\frac{\nu_{q+1}^*}{z_1 - z_1^*} \neq 0, \quad i = 1, \ldots, q\,.$$

The number $\kappa_-\big(\widetilde{G}\big)$ *of negative eigenvalues of* G *is given by*

$$\kappa_-\big(\widetilde{G}\big) = \begin{cases} q & \text{if} \quad \operatorname{Im}\nu_0 > 0\,, \\ q + 1 & \text{if} \quad \operatorname{Im}\nu_0 < 0\,. \end{cases}$$

Proof. By (4.4), $\langle u, u \rangle_{\mathcal{P}} = \nu_1$. The assumptions and the formula (4.3) for ν_i imply the equalities

$$\big\langle (A - z_1)^{-j} u, (A - z_1)^{-i} u \big\rangle_{\mathcal{P}} = 0, \quad 0 \leq i, j \leq q,\ (i, j) \neq (0, 0)\,, \tag{6.2}$$

and

$$\big\langle (A - z_1)^{-q-1} u, u \big\rangle_{\mathcal{P}} = \frac{\nu_{q+1}}{z_1 - z_1^*} \neq 0\,. \tag{6.3}$$

By taking adjoints we see that the equalities (6.2) remain true if we replace z_1 by z_1^*. These formulas imply that the Gram matrix $\widetilde{G}$ and the $q \times q$ matrix G have the asserted forms. The formula for $\kappa_-(\widetilde{G})$ follows immediately from the structure of the matrix $\widetilde{G}$. □

Lemma 6.2. *Assume* $\operatorname{Im}\nu_0 \neq 0$ *and that* (6.1) *holds, and set* $u = \varphi(z_1)$*. Then* z_1 *is an eigenvalue of the self-adjoint relation* $\widehat{A}$ *in* $\widehat{\mathcal{P}}$ *in the minimal self-adjoint realization of* $\widehat{n}(z)$ *with eigenelement* $(A - z_1)^{-1} u$*, the* k *elements*

$$(A - z_1)^{-1} u,\ (A - z_1)^{-2} u, \ldots, (A - z_1)^{-q} u$$

form a maximal Jordan chain of $\widehat{A}$ *at* z_1*, and they span the rootspace of* $\widehat{A}$ *at* z_1*. These statements also hold with* z_1 *replaced by* z_1^**.*

Proof. Set

$$\mathcal{L}_{z_1} = \operatorname{span}\big\{(A - z_1)^{-1} u,\ (A - z_1)^{-2} u, \ldots, (A - z_1)^{-q} u\big\}\,. \tag{6.4}$$

For $j = 1, 2, \ldots$,

$$\big\{(A - z_1)^{-j} u, (A - z_1)^{-j+1} u + z_1 (A - z_1)^{-j} u\big\} \in A\,.$$

Since, by Lemma 6.1, $\mathcal{L}_{z_1} \subset \widehat{\mathcal{P}}$, $\widehat{P}\big(u + z_1 (A - z_1)^{-1}\big) = z_1 (A - z_1)^{-1}$, and for $j = 2, \ldots, q$, the element $(A - z_1)^{-j+1} u + z_1 (A - z_1)^{-j} u$ belongs to $\widehat{\mathcal{P}}$ we have that

$$\big\{(A - z_1)^{-1} u,\ z_1 (A - z_1)^{-1} u\big\} \in \widehat{A}$$

and

$$\big\{(A - z_1)^{-j} u, (A - z_1)^{-j+1} u + z_1 (A - z_1)^{-j} u\big\} \in \widehat{A}, \quad j = 2, \ldots, q\,.$$

Hence z_1 is an eigenvalue of $\widehat{A}$ and the elements which span $\mathcal{L}_{z_1}$ in (6.4) form a chain. This chain is maximal. Indeed, if it would not be maximal then there exist elements $v \in \widehat{\mathcal{P}}$ and $w \in \mathcal{P}$ such that $\{v, w\} \in A$ and $\widehat{P}w = (A - z_1)^{-q}u + z_1 v$ and hence

$$w - z_1 v - \frac{\langle w, u\rangle_{\mathcal{P}}}{\langle u, u\rangle_{\mathcal{P}}} u = (A - z_1)^{-q} u\,.$$

Applying $(A - z_1)^{-1}$ to both sides we obtain

$$(A - z_1)^{-q-1} u = v - \frac{\langle w, u\rangle_{\mathcal{P}}}{\langle u, u\rangle_{\mathcal{P}}} (A - z_1)^{-1} u \in \widehat{\mathcal{P}}\,,$$

which cannot be true because of (6.3). This proves the statements related to z_1. The statements related to z_1^* follow from those for z_1 and the spectral properties of self-adjoint relations, but can also be shown directly by replacing z_1 in the above by z_1^*. $\square$

Theorem 6.3. *Assume $n(z) \in \mathbf{N}$ has Taylor expansion* (2.1) *at z_1 with* $\operatorname{Im}\nu_0 \neq 0$, *$\widehat{n}(z)$ is defined and $\not\equiv \infty$, and $\widehat{n}(z)$ has a pole of order q at z_1. Then $\widetilde{n}(z) = [\widehat{n}]^{\wedge}(z)$ is defined and its minimal self-adjoint realization $\big(\widetilde{\mathcal{P}}, \widetilde{A}, \widetilde{\varphi}(z)\big)$ is given by*

$$\widetilde{\mathcal{P}} = \mathcal{P} \ominus \mathcal{L}, \quad \widetilde{A} = \widetilde{P}A\big|_{\widetilde{\mathcal{P}}}\,,$$

where, with $u = \varphi(z_1)$, $\mathcal{L}$ is the nondegenerate subspace

$$\mathcal{L} = \operatorname{span}\big\{u, (A - z_1)^{-1}u, \ldots, (A - z_1)^{-q}u, (A - z_1^*)^{-1}u, \ldots, (A - z_1^*)^{-q}u\big\}$$

of $\mathcal{P}$, and $\widetilde{P}$ is the orthogonal projection in $\mathcal{P}$ onto $\widetilde{\mathcal{P}}$, and

$$\begin{aligned}\widetilde{\varphi}(z) &= \frac{1}{(z - z_1)} \frac{\langle u, u\rangle_{\mathcal{P}}}{\big\langle (A - z)^{-1}u, u\big\rangle_{\mathcal{P}}} \widetilde{P}\varphi(z)\\ &= \frac{\langle u, u\rangle_{\mathcal{P}}}{\big\langle (A - z)^{-1}(A - z_1)^{-q}u, u\big\rangle_{\mathcal{P}}} \widetilde{P}(A - z)^{-1}(A - z_1)^{-q}u\,.\end{aligned}$$

Moreover, $\widetilde{\mathcal{P}}$ is a Pontryagin space with negative index

$$\operatorname{ind}_-(\mathcal{P}) = \begin{cases} \operatorname{ind}_-(\mathcal{P}) - q & \text{if} \quad \operatorname{Im}\nu_0 > 0\,,\\ \operatorname{ind}_-(\mathcal{P}) - q - 1 & \text{if} \quad \operatorname{Im}\nu_0 < 0\,.\end{cases}$$

The second equality in the formula for $\widetilde{\varphi}(z)$ follows from the expansion

$$(A - z)^{-1} = (z - z_1)^q (A - z)^{-1}(A - z_1)^{-q} + \sum_{j=1}^{q} (z - z_1)^{j-1}(A - z_1)^{-j}$$

and the formulas $\varphi(z) = u + (z - z_1)(A - z)^{-1}u$,

$$\widetilde{P}(A - z_1)^{-j}u = 0, \quad j = 0, 1, \ldots, q, \quad \big\langle (A - z_1)^{-j}u, u\big\rangle_{\mathcal{P}} = 0, \quad j = 1, 2, \ldots, q\,.$$

Moreover, from (6.3) we obtain

$$\widetilde{\varphi}(z_1) = \frac{\langle u, u\rangle_{\mathcal{P}}}{\big\langle (A - z_1)^{-q-1}u, u\big\rangle_{\mathcal{P}}} \widetilde{P}\,(A - z_1)^{-q-1}u = \frac{\nu_0 - \nu_0^*}{\nu_{q+1}} \widetilde{P}\,(A - z_1)^{-q-1}u\,,$$

and this readily implies the following corollary.

Corollary 6.4. *The minimal self-adjoint realization of* $\widetilde{n}(z)$ *in Theorem* 6.3 *centered at* z_1 *is given by*

$$\widetilde{n}(z) = \widetilde{n}(z_1)^* + (z - z_1^*)\big\langle\big(I + (z - z_1)(\widetilde{A} - z)^{-1}\big)\widetilde{u}, \widetilde{u}\big\rangle_{\widetilde{\mathcal{P}}},$$

where

$$\widetilde{u} = \frac{\nu_0 - \nu_0^*}{\nu_{q+1}}\,\widetilde{P}\,(A - z_1)^{-q-1}u\,.$$

Proof of Theorem 6.3. By Theorem 4.2,

$$\widehat{n}(z) \sim \big(\widehat{\mathcal{P}}, \widehat{A}, \widehat{\varphi}(z)\big)$$

with $\widehat{\mathcal{P}} = \mathcal{P} \ominus \operatorname{span}\{u\}$, $\widehat{A} = \widehat{P}A\big|_{\widehat{\mathcal{P}}}$, where $\widehat{P}$ is the orthogonal projection in $\mathcal{P}$ onto $\widehat{\mathcal{P}}$, and

$$\widehat{\varphi}(z) = \frac{1}{z - z_1}\,\frac{\langle u, u\rangle_{\mathcal{P}}}{\big\langle (A - z)^{-1}u, u\big\rangle_{\mathcal{P}}}\,\widehat{P}\,\varphi(z)\,.$$

We apply Theorem 4.4 together with Lemmas 6.1 and 6.2 and we find that the ingredients of the minimal self-adjoint realization of $\widetilde{n}(z)$ are given by

$$\widetilde{\mathcal{P}} = \widehat{\mathcal{P}} \ominus (\mathcal{L}_{z_1} \dotplus \mathcal{L}_{z_1^*}) = \mathcal{P} \ominus \mathcal{L}\,,$$

where $\mathcal{L}_{z_1}$ and $\mathcal{L}_{z_1^*}$ stand for the root spaces of $\widehat{A}$ at z_1 and z_1^*, respectively, $\widetilde{A}$ is the restriction of $\widehat{A}$ to its invariant subspace $\widehat{\mathcal{P}} \ominus (\mathcal{L}_{z_1} \dotplus \mathcal{L}_{z_1^*})$ and hence

$$\widetilde{A} = \widehat{A}\big|_{\widehat{\mathcal{P}} \ominus (\mathcal{L}_{z_1} \dotplus \mathcal{L}_{z_1^*})} = \widetilde{P}A\big|_{\widetilde{\mathcal{P}}}\,.$$

Finally,

$$\widetilde{\varphi}(z) = \widehat{P}_2\,\widehat{\varphi}(z) = \frac{1}{(z - z_1)}\,\frac{\langle u, u\rangle_{\mathcal{P}}}{\big\langle (A - z)^{-1}u, u\big\rangle_{\mathcal{P}}}\,\widetilde{P}\,\varphi(z)\,,$$

where $\widehat{P}_2$ stands for the projection in $\widehat{\mathcal{P}}$ onto $\widehat{\mathcal{P}} \ominus (\mathcal{L}_{z_1} \dotplus \mathcal{L}_{z_1^*})$, and hence $\widehat{P}\widehat{P}_2 = \widetilde{P}$. The formula for $\operatorname{ind}_-(\widetilde{P})$ follows from the last relation in Lemma 6.1. $\square$

Case $\operatorname{Im}\nu_0 = 0$. Let again k be the smallest integer ≥ 1 such that $\nu_k \neq 0$. We recall that such a k exists because we assume that $n(z)$ is not a linear function and, in particular, not a real constant. We also recall (see Remark 2.3 (1)) that $\widehat{n}(z)$ has a pole if and only if $a_k = b_k$. In this case the order of the pole is $q > 0$ if and only if we have

$$n(z) - \alpha(z) = c_q(z - z_1)^{2k+q} + \mathrm{O}\big((z - z_1)^{2k+q+1}\big), \quad c_q \neq 0\,. \tag{6.5}$$

This q is finite, because we assume that $n(z)$ is not linear, that is, $n(z) \not\equiv \alpha(z)$.

Theorem 6.5. *Assume* $n(z) \in \mathbf{N}$ *has Taylor expansion* (2.1) *at* z_1 *with* $\operatorname{Im}\nu_0 = 0$ *and let* $k \geq 1$ *be the smallest integer such that* $\nu_k \neq 0$. *Assume also that* $\widehat{n}(z)$ *is defined and has a pole of order* $q > 0$ *at* z_1, *that is,* (6.5) *holds. Then the minimal self-adjoint realization* $\big(\widetilde{\mathcal{P}}, \widetilde{A}, \widetilde{\varphi}(z)\big)$ *of* $\widetilde{n}(z)$ *is given by*

$$\widetilde{\mathcal{P}} = \mathcal{P} \ominus \mathcal{L}_q, \quad \widetilde{A} = \widetilde{P}A\big|_{\widetilde{\mathcal{P}}}\,,$$

where, with $u = \varphi(z_1)$, $\mathcal{L}_q$ *is the nondegenerate subspace*

$$\mathcal{L}_q = \operatorname{span}\Big\{u, (A - z_1)^{-1}u, \dots, (A - z_1)^{-k-q+1}u, (A - z_1^*)^{-1}u, \dots, (A - z_1^*)^{-k-q}u\Big\}$$

of $\mathcal{P}$, $\widetilde{P}$ *is the orthogonal projection in* $\mathcal{P}$ *onto* $\widetilde{\mathcal{P}}$, *and*

$$\widetilde{\varphi}(z) = \frac{\widehat{n}(z) - \nu_0}{n(z) - \nu_0}\widetilde{P}\varphi(z) = \frac{\widehat{n}(z) - \nu_0}{n(z) - \nu_0}(z - z_1)^{k+q}\widetilde{P}(A - z)^{-1}(A - z_1)^{-k-q+1}u\,. \quad (6.6)$$

Moreover, $\widetilde{\mathcal{P}}$ *is a Pontryagin space with negative index*

$$\kappa_-(\widetilde{\mathcal{P}}) = \kappa_-(\mathcal{P}) - k - q\,. \quad (6.7)$$

The following result can be deduced from Theorem 6.5 in the same way as Corollary 5.3 was derived from Theorem 5.2; see (5.8) and (5.9). Using (6.5) we find that $\widetilde{\varphi}(z)$ in (6.6) is holomorphic at z_1 and that

$$\widetilde{\varphi}(z_1) = -\frac{\nu_k}{c_q}\,\widetilde{P}\,(A - z_1)^{-k-q}u\,.$$

Corollary 6.6. *The minimal self-adjoint realization of* $\widetilde{n}(z)$ *in Theorem* 6.5 *centered at* z_1 *is given by*

$$\widetilde{n}(z) = \widetilde{n}(z_1)^* + (z - z_1^*)\big\langle\big(I + (z - z_1)(\widetilde{A} - z)^{-1}\big)\widetilde{u}, \widetilde{u}\big\rangle_{\widetilde{\mathcal{P}}}\,,$$

where

$$\widetilde{u} = \frac{\nu_k}{c_q}\,\widetilde{P}(A - z_1)^{-k-q}u, \quad \widetilde{n}(z_1) = \lim_{z \to z_1}\big(\widehat{n}(z) - \widehat{h}_{z_1}(z)\big) - \widehat{h}_{z_1^*}(z_1)\,,$$

where $\widehat{h}_{z_1}(z)$ *and* $\widehat{h}_{z_1^*}(z) = \widehat{h}_{z_1}(z^*)^*$ *are the principal parts of the Laurent expansions of* $\widehat{n}(z)$ *at* z_1 *and* z_1^*.

Proof of Theorem 6.5. On account of Theorem 5.2 and Theorem 4.4 we have

$$\widetilde{\mathcal{P}} = \widehat{\mathcal{P}} \ominus (\mathcal{M}_{z_1} \dotplus \mathcal{M}_{z_1^*}) = \mathcal{P} \ominus \big(\mathcal{L} \oplus (\mathcal{M}_{z_1} \dotplus \mathcal{M}_{z_1^*})\big)\,,$$

where

$$\mathcal{L} = \operatorname{span}\Big\{u, (A - z_1)^{-1}u, \dots, (A - z_1)^{-k+1}u,\\ (A - z_1^*)^{-1}u, (A - z_1^*)^{-2}u, \dots, (A - z_1^*)^{-k}u\Big\}\,,$$

and $\mathcal{M}_{z_1}$ and $\mathcal{M}_{z_1^*}$ are the root subspaces at z_1 and z_1^* of the self-adjoint relation $\widehat{A}$ in the space $\widehat{\mathcal{P}}$ in the minimal self-adjoint realization of $\widehat{n}(z)$, $\widetilde{A}$ is the restriction of $\widehat{A}$ to its invariant subspace $\widetilde{\mathcal{P}}$, that is,

$$\widetilde{A} = \widehat{A}|_{\widetilde{\mathcal{P}}} = \big(\widehat{P}A|_{\widehat{\mathcal{P}}}\big)|_{\widetilde{\mathcal{P}}} = \widetilde{P}A|_{\widetilde{\mathcal{P}}}\,,$$

and, finally,

$$\widetilde{\varphi}(z) = \widehat{P}_1\widehat{\varphi}(z) = \frac{\widehat{n}(z) - \nu_0}{n(z) - \nu_0}\,\widetilde{P}\,\varphi(z)\,,$$

where $\widehat{P}_1$ is the orthogonal projection in $\widehat{\mathcal{P}}$ onto $\widetilde{\mathcal{P}}$ and hence $\widehat{P}_1\widehat{P} = \widetilde{P}$.

We now use some of the notations and results from the proof of Theorem 5.2. We have (see (5.10) and (5.12))

$$[\widehat{n}(z)-\nu_0]_i = \big[\,[n(z)-\nu_0]_i\,\big]^{\widehat{\ }} \sim \big(\widehat{\mathcal{P}}, \widehat{A_i}, \widehat{\varphi}_i(z)\big)\,.$$

From Lemma 5.1 (ii) it follows that

$$\mathcal{M}_w = \operatorname{span}\left\{\widehat{\varphi}_i(w), \frac{\widehat{\varphi}_i'(w)}{1!}, \dots, \frac{\widehat{\varphi}_i^{(q-1)}(w)}{(q-1)!}\right\}, \quad w \in \{z_1, z_1^*\}\,.$$

To calculate the derivatives we use (5.13):

$$\widehat{\varphi}_i(z) = \frac{1}{n(z)-\nu_0}\widehat{P}\varphi(z)\,.$$

Inserting the expansion of $\varphi(z)$ at z_1, that is,

$$\varphi(z) = (z-z_1)^k(A-z)^{-1}(A-z_1)^{-k+1}\varphi(z_1) + \sum_{j=1}^{k}(z-z_1)^{j-1}(A-z_1)^{-j+1}\varphi(z_1)$$

and using $\widehat{P}(A-z_1)^{j-1}\varphi(z_1) = 0$ for $j = 1, 2, \dots, k$, we get

$$\widehat{\varphi}_i(z) = \frac{(z-z_1)^k}{n(z)-\nu_0}\,\widehat{P}\,(A-z)^{-1}(A-z_1)^{-k+1}\varphi(z_1)\,. \tag{6.8}$$

Since the factor $\frac{(z-z_1)^k}{n(z)-\nu_0}$ is holomorphic and nonzero at z_1 this implies that

$$\mathcal{M}_{z_1} = \widehat{P}\ \operatorname{span}\Big\{(A-z_1)^{-k}\varphi(z_1), (A-z_1)^{-k-1}\varphi(z_1), \dots, (A-z_1)^{-k-q+1}\varphi(z_1)\Big\}\,.$$

Similarly, by expanding $\varphi(z)$ at z_1^* we find

$$\widehat{\varphi}_i(z) = \frac{(z-z_1^*)^k}{n(z)-\nu_0}\,\widehat{P}\,(A-z)^{-1}(A-z_1^*)^{-k+1}\varphi(z_1^*)\,,$$

and hence

$$\mathcal{M}_{z_1^*} = \widehat{P}_1\,\operatorname{span}\Big\{(A-z_1^*)^{-k}\varphi(z_1^*), (A-z_1^*)^{-k-1}\varphi(z_1^*), \dots, (A-z_1^*)^{-k-q+1}\varphi(z_1^*)\Big\}\,.$$

Using $u = \varphi(z_1)$ and $\varphi(z_1^*) = u + (z_1^*-z_1)(A-z_1^*)^{-1}u$ we obtain

$$\mathcal{L} \oplus (\mathcal{M}_{z_1} \dotplus \mathcal{M}_{z_1^*}) = \mathcal{L}_q\,.$$

This proves the formula in the theorem for the space $\widetilde{\mathcal{P}}$. That its negative index is given by (6.7) follows from

$$\operatorname{ind}_-(\widehat{\mathcal{P}}) = \operatorname{ind}_-(\mathcal{P}) - k, \quad \operatorname{ind}_-\big(\mathcal{M}_{z_1} \dotplus \mathcal{M}_{z_1^*}\big) = q\,.$$

Further, inserting in (6.8) the expansion

$$(A-z)^{-1} = (A-z_1)^{-1} + (z-z_1)(A-z_1)^{-2} + \dots + (z-z_1)^q(A-z)^{-1}(A-z_1)^{-q+1}$$

and applying the projection $\widehat{P}_1$ to both sides in the resulting equality we get

$$\widehat{P}_1\widehat{\varphi}_i(z) = \frac{(z-z_1)^{k+q}}{n(z)-\nu_0}\,\widetilde{P}(A-z)^{-1}(A-z_1)^{-k-q+1}\varphi(z_1)\,.$$

Substituting this expression in

$$\widetilde{\varphi}(z) = \big(\widehat{n}(z) - \nu_0\big)\widehat{P}_1\widehat{\varphi}_i(z)$$

we arrive at the second equality in (6.6). The minimality of the realization of $\widetilde{n}(z)$ follows from the first equality in (6.6) and the minimality of $\varphi(z)$. Finally, the formula for the negative index of $\widetilde{\mathcal{P}}$ follows from

$$\operatorname{ind}_-\big(\mathcal{L} \oplus (\mathcal{M}_{z_1} \dot{+} \mathcal{M}_{z_1^*})\big) = \operatorname{ind}_-(\mathcal{L}) + \operatorname{ind}_-\big(\mathcal{M}_{z_1} \dot{+} \mathcal{M}_{z_1^*}\big) = k + q\,. \quad \square$$

7. The Schur algorithm

Let $n(z) = n_0(z)$ be a generalized Nevanlinna function which is not a real constant. Then the Schur transform $\widehat{n}(z) = n_1(z)$ is defined. If $n_1(z)$ is not a real constant or ∞, its Schur transform $\widehat{n_1}(z) = n_2(z)$ is defined, and so on. By this procedure we obtain either an infinite sequence of generalized Nevanlinna functions $n_j(z)$, $j = 0, 1, \ldots$, or this procedure stops with some index j_0, which means that $n_{j_0}(z)$ is either a real constant of ∞. We call this repeated application of the Schur transform the *Schur algorithm*. We denote this finite or infinite *Schur sequence*

$$n_0(z) = n(z), n_1(z) = \widehat{n}_0(z), \ldots,\; n_j(z) = \widehat{n}_{j-1}(z), \ldots\,, \tag{7.1}$$

by $(n_j(z))_{j\geq 0}$.

Theorem 7.1. *If $n(z) \in \mathbf{N}$ is not a real constant and the Schur algorithm applied to $n(z)$ gives an infinite sequence $\big(n_j(z)\big)_{j\geq 0}$, then there exists an index j_0 such that $n_j(z) \in \mathbf{N}_0$ for all $j \geq j_0$.*

In the proof of the theorem we use two lemmas which we prove first.

Lemma 7.2. *Assume $n(z) \in \mathbf{N}_\kappa$ is holomorphic at z_1 and has minimal self-adjoint realization (4.1): $n(z) \sim \big(\mathcal{P}, A, \varphi(z)\big)$. Then the kernel $L_n(z,w)$ is holomorphic in z and in w^* at $z = w = z_1$ with Taylor expansion*

$$L_n(z,w) = \frac{n(z) - n(w)^*}{z - w^*} = \sum_{i,j=0}^{\infty} \gamma_{ij}(z - z_1)^i (w - z_1)^{*j}$$

with coefficients

$$\gamma_{00} = \frac{\nu_0 - \nu_0^*}{z_1 - z_1^*}, \quad \gamma_{ij} = \big\langle (A - z_1)^{-j}u, (A - z_1)^{-i}u\big\rangle_{\mathcal{P}},\ i + j > 0\,,$$

where $\nu_0 = n(z_1)$ and $u = \varphi(z_1)$. The coefficients have the following property: There is an integer $m_0 \geq 1$ such all matrices $\big(\gamma_{ij}\big)_{i,j=0}^{m-1}$, $m \geq m_0$, have exactly κ negative eigenvalues counting multiplicities.

Proof. The expansion follows from expanding the expression on the righthand side of the relation

$$L_n(z,w) = \big\langle \varphi(z), \varphi(w)\big\rangle, \quad \varphi(z) = u + (z - z_1)(A - z)^{-1}u\,.$$

The property of the coefficients is but a special case of a more general result from [3] which states that if a kernel $K(z, w)$ has a Taylor expansion in the variables z and w^* around $z = w = z_1$, and if the kernel has κ negative squares, then the coefficients in this expansion have the asserted property. □

In the definition of the class $\mathbf{N}_\kappa$, the index κ is the number of negative squares of the kernel $L_n(z, w)$ on $\mathrm{hol}\,(n)$, which has components in $\mathbb{C}_+$ and in $\mathbb{C}_-$. The following lemma shows that the kernel $L_n(z, w)$ has already κ negative squares on one the components.

Lemma 7.3. *If $n(z) \in \mathbf{N}_\kappa$ then on $\mathrm{hol}\,(n) \cap \mathbb{C}^+$ the kernel $L_n(z, w)$ has κ negative squares.*

Proof. By Cayley transformations over the argument z and the values $n(z)$ the statement can be reduced to the situation of a generalized Schur function: if the Schur kernel of a generalized Schur function $s(\zeta)$ has κ negative squares on the open unit disc $\mathbb{D}$, then also the Schur kernel of the symmetric extension of this function to $\left(\mathbb{C} \setminus \mathbb{T}\right) \cap \mathrm{hol}\,(s)$ has κ negative squares. This was proved in [10, pp. 352–4]. □

Proof of Theorem 7.1. First we consider a function $n(z) \in \mathbb{N}_\kappa$, to which the Schur transformation can be applied infinitely often, and which is such that always Case I with $\mathrm{Im}\,\nu_0 > 0$ applies. Then $\kappa = 0$. To see this, we observe that, according to Theorem 4.2 and Corollary 4.3, for each integer $\ell \geq 0$ the space

$$\mathrm{span}\left\{u, (A - z_1)^{-1}u, (A - z_1)^{-2}u, \ldots, (A - z_1)^{-\ell}u\right\}$$

(whose orthogonal complement is the state space in the minimal self-adjoint realization of $n_{\ell+1}(z)$) is a positive subspace. According to Lemma 7.2 the kernel $L_n(z, w)$ is positive definite on $\mathbb{C}^+ \cap \mathrm{hol}\,(n)$, and hence according to Lemma 7.3 it is positive definite. But this means that $\kappa = 0$.

The Schur transformation in Case I with $\mathrm{Im}\,\nu_0 < 0$, in Case II and also in Case III reduces the number of negative squares. Therefore there exists an index j_0 such that starting from n_{j_0} only Case I with $\mathrm{Im}\,\nu_0 > 0$ of the Schur transformation applies. By the first part of this proof, for $j \geq j_0$, $n_j(z)$ is a Nevanlinna function. □

We give an example which shows the dependence of j_0 on z_1.

Example 7.4. We consider the function as in Example 2.4. Assume $\alpha_0, \beta_0, \gamma_0 \in \mathbb{R}$ with $\beta_0 < 0$ and $\gamma_0 > 0$. Then $n_0(z)$ determined by relation

$$n_0(z) = \alpha_0 + \beta_0 z + \mathrm{i}\gamma_0, \quad \mathrm{Im}\, z > 0\,,$$

belongs to $\mathbf{N}_1$. Using induction we find that if

$$\gamma_0 \neq -\ell\beta_0 \mathrm{Im} z_1, \quad \ell = 1, 2, \ldots\,,$$

then for all $j = 1, 2, \ldots,$

$$n_j(z) = \widehat{n}_{j-1}(z) = \alpha_j + \beta_j z + \mathrm{i}\gamma_j, \quad \mathrm{Im}\, z > 0\,, \tag{7.2}$$

where

$$\alpha_j = \alpha_0 - j\,\frac{\beta_0^2 \operatorname{Im} z_1 \operatorname{Re} z_1}{}\gamma_0, \quad \beta_j = \beta_0\left(1 + j\,\frac{\beta_0 \operatorname{Im} z_1}{\gamma_0}\right), \quad \gamma_j = \gamma_0\left(1 + j\,\frac{\beta_0 \operatorname{Im} z_1}{\gamma_0}\right)^2$$

and

$$\operatorname{Im} \widehat{n}_j(z_1) = \gamma_0\left(1 + j\,\frac{\beta_0 \operatorname{Im} z_1}{\gamma_0}\right)\left(1 + (j+1)\frac{\beta_0 \operatorname{Im} z_1}{\gamma_0}\right).$$

It follows that if for some integer $j_0 \geq 1$,

$$-(j_0 - 1)\beta_0 \operatorname{Im} z_1 < \gamma_0 < -j_0\beta_0 \operatorname{Im} z_1\,,$$

then $n_j(z) \in \mathbf{N}_1$ for all $j = 0, \ldots, j_0 - 1$ and $n_j(z) \in \mathbf{N}_0$ for all $j \geq j_0$.

On the other hand if for some integer $j_0 \geq 1$ we have

$$\gamma_0 = -j_0\,\beta_0 \operatorname{Im} z_1\,,$$

then (7.2) holds for all $j = 0, \ldots, j_0 - 1$, and these functions belong to $\mathbf{N}_1$. But we have that

$$n_{j_0}(z) = \alpha_{j_0} + \beta_{j_0} z + \mathrm{i}\gamma_{j_0}, \quad \operatorname{Im} z > 0\,,$$

where

$$\begin{aligned} \alpha_{j_0} &= \alpha_{j_0-1} - \frac{\beta_{j_0-1}^2 \operatorname{Im} z_1 \operatorname{Re} z_1}{\gamma_{j_0-1}} = \alpha_0 + \frac{j_0+1}{j_0}\beta_0 \operatorname{Re} z_1\,, \\ \beta_{j_0} &= -\beta_{j_0-1} = -\beta_0\left(1 + (j_0-1)\frac{\beta_0 \operatorname{Im} z_1}{\gamma_0}\right) = -\frac{\beta_0}{j_0}\,, \end{aligned}$$

and

$$\gamma_{j_0} = \gamma_{j_0-1} = \gamma_0\left(1 + (j_0-1)\frac{\beta_0 \operatorname{Im} z_1}{\gamma_0}\right)^2 = \frac{\gamma_0}{j_0^2}\,.$$

Now $n_{j_0}(z)$ belongs to $\mathbf{N}_0$ and if we continue the Schur algorithm so do all subsequent Schur transforms. Thus in this example $j_0 = -\gamma_0/(\beta_0 \operatorname{Im} z_1)$ if the number on the righthand side an integer, otherwise j_0 is the integral part of $-\gamma_0/(\beta_0 \operatorname{Im} z_1)$ plus 1 and hence increases when $z_1 \in \mathbb{C}^+$ moves closer to the real axis.

Remarks 7.5. (1) If $n(z) \in \mathbf{N}$ and the Schur algorithm can be applied to it to yield the Schur sequence $(n_j(z))_{j\geq 0}$ then

$$\dim \mathcal{P}_{j+1} < \dim \mathcal{P}_j,$$

where $\mathcal{P}_j$ is the state space in the self-adjoint realization of $n_j(z)$. This follows from Theorems 4.2, 4.4, and 5.2. Hence the Schur sequence is finite if and only if $n(z)$ is rational or, equivalently, the reproducing kernel Pontryagin space $\mathcal{L}(n)$ with kernel $L_n(z,w)$ is finite dimensional.

(2) Now assume that the Schur sequence is infinite and

$$n(z) \sim \big(\mathcal{P}, A, \varphi(z)\big); \quad n_j(z) \sim \big(\mathcal{P}_j, A_j, \varphi_j(z)\big), \quad j = 1, 2, \ldots.$$

If $n(z)$ and $n_j(z)$ are holomorphic at z_1 there exist nonnegative integers ℓ_j and m_j such that the entries of the row vector

$$\mathfrak{B}_j = \big(u\ (A–z_1)^{-1}u\ \ldots\ (A–z_1)^{-\ell_j}u\ \ (A–z_1^*)^{-1}u\ (A–z_1^*)^{-2}u\ \ldots\ (A–z_1^*)^{-m_j}u\big)\,,$$

where $u = \varphi(z_1)$, are linearly independent and span a nondegenerate subspace of $\mathcal{P}$ and

$$\mathcal{P}_j = \mathcal{P} \ominus \operatorname{span}\{\mathfrak{B}_j\}, \quad A_j = P_j A\big|_{\mathcal{P}_j}, \quad \varphi_j(z) = \rho_j(z) P_j \varphi(z)\,,$$

where P_j is the orthogonal projection in $\mathcal{P}$ onto $\mathcal{P}_j$ and $\rho_j(z)$ is a nonvanishing holormorphic function on $\operatorname{hol}(n) \cap \operatorname{hol}(n_j)$. Moreover $\ell_j \to \infty$ whereas m_j remains bounded as $j \to \infty$ and the resolvent of A_j is given by (5.15). These results follow from applying the theorems in Sections 4–6 and using induction.

8. The effect of the Schur transformation on the difference quotient operator

In the realization (4.1) of $n(z) \in \mathbf{N}$, the self-adjoint relation A may be unbounded. In the setting of the reproducing kernel Pontryagin space $\mathcal{L}(n)$ with kernel

$$L_n(z,w) = \frac{n(z) - n(w)^*}{z - w^*}$$

it is natural to study the effect of the Schur transformation on the bounded difference quotient operator R_z, which coincides with the resolvent operator of A. Recall that $\mathcal{U}_\ell$ is the class of all rational J_ℓ-unitary 2×2 matrix functions. For a function $\Theta(z)$ in this class, $\mathcal{P}(\Theta)$ is the reproducing kernel Pontryagin space with kernel

$$K_\Theta(z,w) = \frac{J_\ell - \Theta(z) J_\ell \Theta(w)^*}{z - w^*}, \quad J_\ell = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The elements of $\mathcal{P}(\Theta)$ are 2-vector functions whereas the function in $\mathcal{L}(n)$ are scalar.

Theorem 8.1. *Let $n(z) \in \mathbf{N}_\kappa$ and suppose that there exists a matrix function*

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \mathcal{U}_\ell$$

such that the mapping

$$u(\zeta) \longrightarrow \begin{pmatrix} 1 & -n(\zeta) \end{pmatrix} u(\zeta)$$

is an isometry from $\mathcal{P}(\Theta)$ into $\mathcal{L}(n)$. Define $\widetilde{n}(z) = \mathcal{T}_{\Theta(z)^{-1}}(n(z))$, that is, $n(z) = \mathcal{T}_{\Theta(z)}(\widetilde{n}(z))$. Then the following statements hold.

(i) *$\widetilde{n}(z) \in \mathbf{N}_{\kappa-\kappa(\Theta)}$, where $\kappa(\Theta)$ is the number of negative squares of the kernel $K_\Theta(z,w)$.*

(ii) *The mapping* $g(\zeta) \mapsto f(\zeta)$*:*

$$f(\zeta) = \big(a(\zeta) - n(\zeta)c(\zeta)\big)g(\zeta)$$

is an isometry from $\mathcal{L}(\widetilde{n})$ *into* $\mathcal{L}(n)$*, and we have*

$$\mathcal{L}(n) = \begin{pmatrix}1 & -n\end{pmatrix}\mathcal{P}(\Theta) \oplus (a - nc)\mathcal{L}(\widetilde{n})\,.$$

In other words, the mapping

$$W : \mathcal{L}(n) \ni f(\zeta) \mapsto \begin{pmatrix} u(\zeta) \\ g(\zeta) \end{pmatrix} \in \begin{pmatrix} \mathcal{K}(\Theta) \\ \mathcal{L}(\widetilde{n}) \end{pmatrix},$$

where $f(\zeta), u(\zeta)$ *and* $g(\zeta)$ *are connected by the relation*

$$f(\zeta) = \begin{pmatrix}1 & -n(\zeta)\end{pmatrix}u(\zeta) + \big(a(\zeta) - n(\zeta)c(\zeta)\big)g(\zeta)\,, \tag{8.1}$$

is a unitary mapping from $\mathcal{L}(n)$ *onto* $\mathcal{K}(\Theta) \oplus \mathcal{L}(\widetilde{n})$*.*

(iii) *The mapping* WR_zW^* *is of the form*

$$WR_zW^* = \begin{pmatrix} R_{11}(z) & R_{12}(z) \\ R_{21}(z) & R_{22}(z) \end{pmatrix} : \begin{pmatrix} \mathcal{P}(\Theta) \\ \mathcal{L}(\widetilde{n}) \end{pmatrix} \to \begin{pmatrix} \mathcal{P}(\Theta) \\ \mathcal{L}(\widetilde{n}) \end{pmatrix}, \tag{8.2}$$

with

$$\begin{aligned}
R_{11}(z) &= R_z - \frac{1}{k(z)}(R_z\Theta)(\,\cdot\,)\begin{pmatrix}\widetilde{n}(z) \\ 1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} E_z \\
&= R_z - K_\Theta(\,\cdot\,, z^*)\begin{pmatrix}1 \\ -n(z)\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} E_z\,, \\
R_{12}(z) &= \frac{1}{k(z)}(R_z\Theta)(\,\cdot\,)\begin{pmatrix}d(z) \\ -c(z)\end{pmatrix} E_z \\
&= -\big(a(z) - n(z)c(z)\big)K_\Theta(\,\cdot\,, z^*)\begin{pmatrix}0 \\ 1\end{pmatrix} E_z, \\
R_{21}(z) &= -\frac{1}{k(z)}(R_z\widetilde{n})(\,\cdot\,)\begin{pmatrix}0 & 1\end{pmatrix} E_z \\
&= -\frac{1}{k(z)}L_{\widetilde{n}}(\,\cdot\,, z^*)\begin{pmatrix}0 & 1\end{pmatrix} E_z\,, \\
R_{22}(z) &= R_z - \frac{c(z)}{k(z)}(R_z\widetilde{n})(\,\cdot\,)\,E_z \\
&= R_z - \frac{c(z)}{k(z)}L_{\widetilde{n}}(\,\cdot\,, z^*)\,E_z\,,
\end{aligned}$$

where R_z *is the difference-quotient operator and* E_z *is the operator of evaluation at the point* z *on any reproducing kernel space and*

$$k(z) = c(z)\widetilde{n}(z) + d(z) = \frac{\det\Theta(z)}{a(z) - n(z)c(z)}\,. \tag{8.3}$$

The assumptions of the theorem hold for $n(z) \in \mathbf{N}_\kappa$ and functions $\Theta(z)$ as in (3.5) (see [2]); in these cases $\widetilde{n}(z)$ in the theorem is the Schur transform $\widehat{n}(z)$ of $n(z)$ defined in Section 2.

Proof of Theorem 8.1. We first prove item (ii) and to do that we use the decomposition

$$L_n(z,w) = \begin{pmatrix}1 & -n(z)\end{pmatrix} K_\Theta(z,w) \begin{pmatrix}1 \\ -n(w)^*\end{pmatrix} + \\ + \big(a(z)-n(z)c(z)\big) L_{\widetilde{n}}(z,w) \big(a(w)-n(w)c(w)\big)^* .$$

By hypothesis the space $\begin{pmatrix}1 & -n\end{pmatrix} \mathcal{P}(\Theta)$ is isometrically included in $\mathcal{L}(n)$ and hence we have the orthogonal decomposition

$$\mathcal{L}(n) = \begin{pmatrix}1 & -n\end{pmatrix} \mathcal{P}(\Theta) \oplus (a-nc)\mathcal{L}(\widetilde{n}) .$$

Assume first that $a(z)-n(z)c(z) \not\equiv 0$. Then the map of multiplication by $(a-nc)$ is an isometry from $\mathcal{L}(\widetilde{n})$ into $\mathcal{L}(n)$; see [4, Theorem 1.5.7]. It follows that W is an isometry because, by assumption, the map of multiplication by $\begin{pmatrix}1 & -n\end{pmatrix}$ is also an isometry from $\mathcal{P}(\Theta)$ into $\mathcal{L}(n)$. This also holds when $a(z)-n(z)c(z) \equiv 0$.

By the unitarity of W, item (i) follows from item (ii).

Now we prove item (iii). From

$$(R_z f)(\zeta) = \begin{cases} \dfrac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \neq z , \\ f'(z) , & \zeta = z , \end{cases}$$

we have

$$\big(R_z(fh)\big)(\zeta) = (R_z f)(\zeta) h(z) + f(\zeta)(R_z h)(\zeta) . \tag{8.4}$$

In the formulas that follow we suppress the dependence on the variable ζ. Using (8.4) we compute the decomposition of R_z on $\mathcal{L}(n)$ starting from (8.1):

$$R_z f = \begin{pmatrix}1 & -n\end{pmatrix} \left\{ R_z u + R_z \Theta \begin{pmatrix}1 \\ 0\end{pmatrix} g(z) + \Theta \begin{pmatrix}1 \\ 0\end{pmatrix} R_z g \right\} \\ - R_z n \left\{ \begin{pmatrix}0 & 1\end{pmatrix} u(z) + c(z) g(z) \right\} .$$

Applying R_z to both sides of the equality $\begin{pmatrix}1 & -n\end{pmatrix} \Theta = (a-nc) \begin{pmatrix}1 & -\widetilde{n}\end{pmatrix}$, we obtain

$$\begin{aligned} &\begin{pmatrix}0 & -R_z n\end{pmatrix} \Theta(z) + \begin{pmatrix}1 & -n\end{pmatrix} R_z \Theta \\ &\qquad = R_z(a-nc) \begin{pmatrix}1 & -\widetilde{n}(z)\end{pmatrix} + (a-nc) \begin{pmatrix}0 & -R_z \widetilde{n}\end{pmatrix} \\ &\qquad = (R_z a - R_z n c(z) - n R_z c) \begin{pmatrix}1 & -\widetilde{n}(z)\end{pmatrix} + (a-nc) \begin{pmatrix}0 & -R_z \widetilde{n}\end{pmatrix} \\ &\qquad = \left(\begin{pmatrix}1 & -n\end{pmatrix} R_z \Theta \begin{pmatrix}1 \\ 0\end{pmatrix} - R_z n c(z) \right) \begin{pmatrix}1 & -\widetilde{n}(z)\end{pmatrix} + (a-nc) \begin{pmatrix}0 & -R_z \widetilde{n}\end{pmatrix} . \end{aligned}$$

Now we multiply both sides by $\begin{pmatrix}0 \\ 1\end{pmatrix}$ and after some calculation we find

$$R_z n = \frac{1}{k(z)} \left((a-nc) R_z \widetilde{n} + \begin{pmatrix}1 & -n\end{pmatrix} R_z \Theta \begin{pmatrix}\widetilde{n}(z) \\ 1\end{pmatrix} \right) .$$

We substitute this into the formula for $R_z f$:

$$\begin{aligned}R_z f = \begin{pmatrix}1 & -n\end{pmatrix} &\left\{R_z - \frac{1}{k(z)} R_z \Theta \begin{pmatrix}\widetilde{n}(z) \\ 1\end{pmatrix} \begin{pmatrix}0 & 1\end{pmatrix} E_z\right\} u \\ + \begin{pmatrix}1 & -n\end{pmatrix} &\left\{\frac{1}{k(z)} R_z \Theta \begin{pmatrix}d(z) \\ -c(z)\end{pmatrix} E_z\right\} g \\ + (a - nc) &\left\{R_z - \frac{c(z)}{k(z)} R_z \widetilde{n} E_z\right\} g \\ - (a - nc) &\left\{\frac{1}{k(z)} R_z \widetilde{n}(z) \begin{pmatrix}0 & 1\end{pmatrix} E_z\right\} u\end{aligned}$$

and hence the first equalities in the formulas for R_{ij}, $i, j = 1, 2$. The second equalities come from these and the formulas:

$$R_z \Theta = K_\Theta(\,\cdot\,, z^*) J_\ell \Theta(z)\,, \tag{8.5}$$

and

$$J_\ell \Theta \begin{pmatrix} d \\ -c \end{pmatrix} = - \begin{pmatrix} 0 \\ \det \Theta \end{pmatrix}, \quad \frac{1}{k} J_\ell \Theta \begin{pmatrix} d \\ -c \end{pmatrix} = - \begin{pmatrix} 0 \\ a - nc \end{pmatrix}, \quad \frac{1}{k} J_\ell \Theta \begin{pmatrix} \widetilde{n} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -n \end{pmatrix}. \quad \square$$

We apply this theorem to give another proof of the formula for $\widehat{u}$ in Corollary 4.3 and the resolvent of $\widehat{A}$ in formula (4.6) in the setting of reproducing kernel spaces. We assume that $n(z)$ is holomorphic at z_1, that $\operatorname{Im} n(z_1) \neq 0$ and that (2.4) holds, that is,

$$\nu_1 \neq \frac{\nu_0 - \nu_0^*}{z_1 - z_1^*}\,.$$

Then $n(z)$ has the minimal self-adjoint realization centered at z_1 given by

$$n(z) = n(z_1)^* + (z - z_1^*)\big\langle\big(I + (z - z_1)R_z\big)u, u\big\rangle_{\mathcal{L}(n)}, \quad u(\zeta) = L_n(\zeta, z_1) \in \mathcal{L}(n)\,, \tag{8.6}$$

and $\widehat{n}(z)$ has the minimal self-adjoint realization centered at z_1 given by

$$\widehat{n}(z) = \widehat{n}(z_1)^* + (z - z_1^*)\big\langle\big(I + (z - z_1)R_z\big)\widehat{u}, \widehat{u}\big\rangle_{\mathcal{L}(\widehat{n})}, \quad \widehat{u}(\zeta) = -L_{\widehat{n}}(\zeta, z_1^*)\,. \tag{8.7}$$

Theorem 8.2. *With the above assumptions on $n(z)$, the realizations* (8.6) *and* (8.7) *for $n(z)$ and its Schur transform $\widehat{n}(z)$ we have*

$$\widehat{u} = \frac{\nu_0 - \nu_0^*}{\nu_1 - \mu} P W R_{z_1} W^* W u\,, \tag{8.8}$$

where P is the orthogonal projection in $\mathcal{P}(\Theta) \oplus \mathcal{L}(\widehat{n})$ onto $\mathcal{L}(\widehat{n})$, and

$$\begin{pmatrix} 0 \\ R_z g \end{pmatrix} = W R_z W^* \begin{pmatrix} 0 \\ g \end{pmatrix} - \frac{\left\langle W R_z W^* \begin{pmatrix} 0 \\ g \end{pmatrix}, W u \right\rangle}{\langle W R_z W^* W u, W u \rangle} W R_z W^* W u, \quad g \in \mathcal{L}(\widehat{n})\,, \tag{8.9}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product in $\mathcal{P}(\Theta) \oplus \mathcal{L}(\widehat{n})$.

Proof. Under the assumptions of the theorem we have $n(z) = \mathcal{T}_{\Theta_1(z)}(\widehat{n}(z))$ with

$$\Theta_1(z) = I_2 + \left(\frac{z - z_1}{z - z_1^*} - 1\right)\frac{\mathbf{u}\mathbf{u}^* J_\ell}{\mathbf{u}^* J_\ell \mathbf{u}}, \quad \mathbf{u} = \begin{pmatrix} \nu_0^* \\ 1 \end{pmatrix}, \quad J_\ell = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

or, in full,

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

with

$$a(z) = 1 + \frac{\nu_0^*}{(z - z_1^*)\mu}, \qquad b(z) = -\frac{|\nu_0|^2}{(z - z_1^*)\mu},$$

$$c(z) = \frac{1}{(z - z_1^*)\mu}, \qquad d(z) = 1 - \frac{\nu_0}{(z - z_1^*)\mu},$$

where

$$\mu = \langle u, u\rangle_{\mathcal{L}(n)} = \frac{\nu_0 - \nu_0^*}{z_1 - z_1^*}.$$

The space $\mathcal{P}(\Theta)$ is 1-dimensional and spanned by

$$F(\zeta) = \frac{\mathbf{u}}{z_1^* - \zeta} = (\nu_0 - \nu_0^*)(R_z\Theta)(\zeta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using formula (8.5):

$$R_z\Theta = K_\Theta(\,\cdot\,, z^*)J_\ell\Theta(z)$$

and the fact that Θ is J_ℓ-unitary, we find that

$$\begin{aligned}
\langle F, F\rangle_{\mathcal{P}(\Theta)} &= |\nu_0 - \nu_0^*|^2 \left\langle R_z\Theta\begin{pmatrix} 1 \\ 0 \end{pmatrix}, R_z\Theta\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\rangle_{\mathcal{P}(\Theta)} \\
&= |\nu_0 - \nu_0^*|^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \Theta(z_1)^* J_\ell^* K_\Theta(z_1^*, z_1^*) J_\ell \Theta(z_1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= |\nu_0 - \nu_0^*|^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \\
&\quad \times \frac{\Theta(z_1)^* J_\ell^* J_\ell J_\ell \Theta(z_1) - \Theta(z_1)^* J_\ell^* \Theta(z_1^*) J_\ell \Theta(z_1^*)^* J_\ell \Theta(z_1)}{z_1^* - z_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= |\nu_0 - \nu_0^*|^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\Theta(z_1)^* J_\ell \Theta(z_1) - J_\ell}{z_1^* - z_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= |\nu_0 - \nu_0^*|^2 \frac{a(z_1)^* c(z_1) - a(z_1) c(z_1)^*}{z_1^* - z_1} \\
&= \mu.
\end{aligned}$$

It follows that

$$K_\Theta(z, w) = F(z)\frac{1}{\mu}F(w)^*, \tag{8.10}$$

because the functions on both sides of the equality are reproducing kernels for the space $\mathcal{P}(\Theta)$. Of course, this formula could also be verified directly.

The mapping

$$u \longrightarrow \begin{pmatrix} 1 & -n \end{pmatrix} u$$

is an isometry from $\mathcal{P}(\Theta)$ into $\mathcal{L}(n)$:

$$\begin{aligned}\big\langle\big(1 \quad -n\big)\, F, \big(1 \quad -n\big)\, F\big\rangle_{\mathcal{L}(N)} &= \big\langle L_n(\,\cdot\,, z_1), L_n(\,\cdot\,, z_1)\big\rangle_{\mathcal{L}(n)} \\ &= L_n(z_1, z_1) = \mu = \langle F, F\rangle_{\mathcal{P}(\Theta)}\,,\end{aligned}$$

and hence we may apply Theorem 8.1.

The function u can be written as

$$u(\zeta) = L_n(\zeta, z_1) = \big(1 \quad -n(\zeta)\big)\, \frac{\mathbf{u}}{z_1^* - \zeta} + \big(a(\zeta) - n(\zeta)c(\zeta)\big)\, 0\,,$$

and hence

$$(Wu)(\zeta) = \begin{pmatrix} \dfrac{\mathbf{u}}{z_1^* - \zeta} \\ 0 \end{pmatrix} = \begin{pmatrix} F(\zeta) \\ 0 \end{pmatrix}.$$

In the inner product of the space $\mathcal{P}(\Theta) \oplus \mathcal{L}(\widehat{n})$, $n(z)$ can also be written as

$$n(z) = n(z_1)^* + (z - z_1^*)\big\langle\big(I + (z - z_1)WR_zW^*\big)Wu, Wu\big\rangle_{\mathcal{P}(\Theta)\oplus\mathcal{L}(\widehat{n})}\,.$$

We now prove formula (8.8). We have

$$\begin{aligned}PWR_zW^*Wu = P\begin{pmatrix} R_{11}(z)F \\ R_{21}(z)F \end{pmatrix} &= R_{21}(z)F = \frac{1}{k(z)(z - z_1^*)} L_{\widehat{n}}(\,\cdot\,, z^*) \\ &= \frac{a(z) - n(z)c(z)}{z - z_1} L_{\widehat{n}}(\,\cdot\,, z^*) = \frac{1}{z - z_1^*}\left(1 - \frac{\nu_0 - n(z)}{\mu(z_1 - z)}\right) L_{\widehat{n}}(\,\cdot\,, z^*)\,.\end{aligned}$$

Here we have used the relation (8.3), $\det\Theta(z) = (z - z_1)/(z - z_1^*)$, and

$$(z - z_1^*)\mu + \nu_0^* = (z - z_1)\mu + \nu_0\,.$$

Now we let $z \to z_1$. Since R_z is holomorphic at z_1 and $L_{\widehat{n}}(\,\cdot\,, z^*) \to L_{\widehat{n}}(\,\cdot\,, z_1^*)$ in $\mathcal{L}(\widehat{n})$ we obtain

$$PWR_{z_1}W^*Wu = \frac{1}{z_1 - z_1^*}\left(1 - \frac{\nu_1}{\mu}\right) L_{\widehat{n}}(\,\cdot\,, z_1^*) = \frac{\nu_1 - \mu}{\nu_0 - \nu_0^*}\widehat{u}\,.$$

This completes the proof of (8.8).

By (8.2), to prove (8.9) it suffices to show that

$$\begin{pmatrix} R_{12}(z)g \\ \big(R_{22}(z) - R_z\big)g \end{pmatrix} = \frac{\left\langle WR_zW^*\begin{pmatrix} 0 \\ g \end{pmatrix}, Wu \right\rangle}{\big\langle WR_zW^*Wu, Wu\big\rangle} WR_zW^*Wu, \quad g \in \mathcal{L}(\widehat{n})\,.$$

Simple substitutions of the relevant formulas yield for the lefthand side:

$$\begin{pmatrix} R_{12}(z)g \\ \big(R_{22}(z) - R_z\big)g \end{pmatrix} = -\begin{pmatrix} K_\Theta(\,\cdot\,, z^*)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \dfrac{1}{(z - z_1)\mu} L_{\widehat{n}}(\,\cdot\,, z^*) \end{pmatrix} \big(a(z) - n(z)c(z)\big)g(z)\,.$$

For the vector function on the righthand side we get

$$W R_z W^* W u = \begin{pmatrix} R_{11} F \\ R_{21} F \end{pmatrix} = \mu \begin{pmatrix} K_\Theta(\,\cdot\,, z^*) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \dfrac{1}{(z - z_1)\mu} L_{\widehat{n}}(\,\cdot\,, z^*) \end{pmatrix} \big(a(z) - n(z)c(z)\big)\,.$$

Indeed, on account of (8.10), the first component in the middle term can be written as

$$\begin{aligned}
\big(R_{11}(z)F\big)(\zeta) &= (R_z F)(\zeta) - K_\Theta(\zeta, z^*) \begin{pmatrix} 1 \\ -n(z) \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} F(z) \\
&= \frac{1}{z_1^* - z} \left(F(\zeta) - F(\zeta) \frac{1}{\mu} F(z^*)^* \begin{pmatrix} 1 \\ -n(z) \end{pmatrix} \right) \\
&= \frac{1}{(z_1^* - z)\mu} F(\zeta) \left(\mu - \frac{\nu_0 - n(z)}{z_1 - z} \right) \\
&= F(\zeta) \frac{1}{\mu} \frac{1}{z_1 - z} \mu \big(a(z) - n(z)c(z)\big) \\
&= \mu F(\zeta) \frac{1}{\mu} F(z^*)^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \big(a(z) - n(z)c(z)\big) \\
&= \mu K_\Theta(\,\cdot\,, z^*) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \big(a(z) - n(z)c(z)\big) \qquad (8.11)
\end{aligned}$$

and the second component as

$$\begin{aligned}
\big(R_{21}(z)F\big)(\zeta) &= -\frac{1}{k(z)} L_{\widehat{n}}(\,\cdot\,, z^*) \begin{pmatrix} 0 & 1 \end{pmatrix} F(z) \\
&= L_{\widehat{n}}(\,\cdot\,, z^*) \frac{a(z) - n(z)c(z)}{z - z_1}\,.
\end{aligned}$$

Thus, to establish (8.9) it suffices to show that

$$-\mu \left\langle W R_z W^* \begin{pmatrix} 0 \\ g \end{pmatrix}, W u \right\rangle = \big\langle W R_z W^* W u, W u \big\rangle g(z), \quad g \in \mathcal{L}(\widehat{n})\,. \qquad (8.12)$$

By the reproducing property of $K_\Theta(z, w)$, the lefthand side is equal to

$$\begin{aligned}
-\mu \left\langle W R_z W^* \begin{pmatrix} 0 \\ g \end{pmatrix}, W u \right\rangle &= -\mu \big\langle R_{12}(z) g, F \big\rangle_{\mathcal{P}(\Theta)} \\
&= \mu \big(a(z) - n(z)c(z)\big) g(z) \left\langle K_\Theta(\,\cdot\,, z^*) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, F(z) \right\rangle_{\mathcal{P}(\Theta)}
\end{aligned}$$

and, by (8.11), the righthand side is equal to

$$\begin{aligned}
\big\langle W R_z W^* W u, W u \big\rangle g(z) &= \big\langle R_{11} F, F \big\rangle_{\mathcal{P}(\Theta)} g(z) \\
&= \mu \big(a(z) - n(z)c(z)\big) g(z) \left\langle K_\Theta(\,\cdot\,, z^*) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, F \right\rangle_{\mathcal{P}(\Theta)}\,.
\end{aligned}$$

Hence the equality (8.12) holds and the proof of (8.9) is complete. □

Theorem 8.2 is the analog of similar results in [1, Theorems 4.1 and 4.2], [13, Theorem 4.2], and [14, Section 6]. These results concern the effect of the Schur transformation on the coisometric and unitary realizations of generalized Schur functions in the setting of reproducing kernel Pontryagin spaces.

References

[1] D. Alpay, T. Y. Azizov, A. Dijksma, H. Langer, and G. Wanjala, *A basic interpolation problem for generalized Schur functions and coisometric realizations*, Operator Theory: Adv., Appl. **143**, Birkhäuser Verlag, Basel, 2003, 39–76.

[2] D. Alpay, A. Dijksma, and H. Langer, *J_ℓ-unitary factorization and the Schur algorithm for Nevanlinna functions in an indefinite setting*, Lin. Algebra Appl. **419** (2–3) (2006), 675–709.

[3] D. Alpay, A. Dijksma, and H. Langer, *The algorithm of Issai Schur and related topics in an indefinite setting*, to appear in the volume *System theory, the Schur algorithm and multidimensional analysis*, in the series *Operator Theory: Advances and Applications*, 2007

[4] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Operator Theory: Adv., Appl. **96**, Birkhäuser Verlag, Basel, 1997.

[5] A. Dijksma, H. Langer, A. Luger, and Y. Shondin, *A factorization result for generalized Nevanlinna functions of the class $\mathcal{N}_\kappa$*, Integral Equations and Operator Theory **36** (2000), 121–125.

[6] A. Dijksma, H. Langer, Y. Shondin, and C. Zeinstra, *Self–adjoint operators with inner singularities and Pontryagin spaces*, Operator Theory: Adv., Appl. **118**, Birkhäuser Verlag, Basel, 2000, 105–175.

[7] A. Dijksma, H. Langer, and H. S. V. de Snoo, *Unitary colligations in Π_κ-spaces, characteristic functions and Straus extensions*, Pac. J. Math. **125** (1986), 347–362.

[8] A. Dijksma, H. Langer, and H. S. V. de Snoo, *Eigenvalues and pole functions of Hamiltonian systems with eigenvalue depending boundary conditions*, Math. Nachr. **161** (1993), 107–154.

[9] T. Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin, 1966.

[10] M. G. Krein and H. Langer, *Über die verallgemeinerte Resolventen und die charakteristische Funktion eines isometrischen Operators in Raume Π_κ*, Colloquia Math. Soc. Janos Bolyai, Tihany (Hungary), 5. Hilbert space operators, 1970, 353–399.

[11] H. Langer and A. Luger, *A class of 2×2-matrix functions*, Glasnik Matematicki. Serija III. Croatian Math. Soc., Zagreb. **34** (55) (2000), 149–160.

[12] W. Stenger, *On the projection of a selfadjoint operator*, Bull. Amer. Math. Soc. **74** (1968), 369–372.

[13] G. Wanjala, *Closely connected unitary realizations of the solutions to the basic interpolation problem for generalized Schur functions*, Operator Theory: Adv., Appl. **160**, Birkhäuser Verlag, Basel, 2005, 439–466.

[14] G. Wanjala, *The Schur transform of a generalized Schur function and operator realizations*, PhD thesis, University of Groningen, The Netherlands, 2005.

Daniel Alpay
Department of Mathematics
Ben-Gurion University of the Negev
P.O. Box 653
IL-84105 Beer-Sheva
Israel
e-mail: dany@math.bgu.ac.il

Aad Dijksma
Department of Mathematics
University of Groningen
P.O. Box 800
NL-9700 AV Groningen
The Netherlands
e-mail: dijksma@math.rug.nl

Heinz Langer
Institute of Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstrasse 8–10
A-1040 Vienna
Austria
e-mail: hlanger@mail.zserv.tuwien.ac.at

Yuri Shondin
Department of Theoretical Physics
Faculty of Mathematics, Informatics and Physics
N. Novgorod State Pedagogical University
Str. Ulyanova 1
RUS-603950 Nizhny Novgorod, GSP 37
Russia
e-mail: shondin@sinn.ru

Communicated by Bernd Kirstein.
Received: September 25, 2006.
Accepted: October 11, 2006.